INFORMATIONAL INERTIA AND THE TAYLOR PRINCIPLE

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ABSTRACT. Determinacy bounds provide the limits on monetary policy's reaction function that rule out self fulfilling equilibria. In standard sticky price analyses, these bounds are generically nonlinear functions of model parameters and all the coefficients in this reaction function. We examine a collection of alternative models with nominal rigidities driven by different informational inertia but all having in common a vertical long Phillips curve. These models share the same determinacy bounds, independent of model specific parameters and dependent only on monetary policy's reaction to inflation. This reaction must be more than one for one - that is, the celebrated Taylor principle is shown to be necessary and not just sufficient if the long run Phillips curve is vertical - no amount of output targeting can substitute for this concern of the monetary authority for inflation.

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1. INTRODUCTION

We provide bounds on monetary policy to deliver a unique equilibrium when the long run Phillips curve is vertical. We find that only the response of monetary policy to inflation matters for uniqueness under this long run condition. Specifically, we derive the bounds in several models of informational inertia with varied short run Phillips curve that, however, are all vertical in the long run. In contrast to the canonical sticky price model, if this long run condition holds, no amount of output gap targeting, forward or backward-looking

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inflation targeting can substitute for a more than one-for-one response to current inflation. That is, the Taylor principle is necessary in an absolute sense.

Our central contribution is to the understanding of the limits on monetary policy to ensure a unique, stationary or determinant equilibrium following Blanchard and Kahn (1980). In particular we cast our attention to models with information rigidities and add the sticky information model of Mankiw and Reis (2002), the imperfect common knowledge model of Nimark (2008) and our general model of finite attentiveness, specified only by imposing the natural rate hypothesis holding at a finite horizon, to the list of models analyzed via restrictions on coefficients in monetary policy's Taylor (1993) rule following Clarida, Galí, and Gertler (2000) and Woodford (2001b) for the sticky price model.[1] We find that a vertical long run Phillips curve brings a unified perspective across models with imperfect information on determinacy and demands that the more than one-for-one response of the nominal interest rate in response to inflation be in response to current inflation directly. This calls assertions such as Woodford (2003, pp. 254–255) "... indeed, a large enough [response to] *either* [the output gap or inflation] suffices to guarantee determinacy" into question. In the absence of a trade off between inflation and the output gap in the long run - a vertical Phillips curve - we show that no degree of output gap targeting can substitute for a reaction to inflation. This has a very similar flavor to Davig and Leeper (2007) - in their model of regime switching monetary policy, it is the long run characteristics that are decisive for determinacy, their long run Taylor principle - we argue the same logic applies to the Phillips curve, its long run slope being the decisive characteristic. We demonstrate the long run verticality of our three Phillips curves and compare to the dynamic trade off inherent in the Calvo model. The lack of a dynamic structure in inflation in the long run in the three inattentive Phillips curves (it is some imperfect forecast or prediction of inflation in inattentive Phillips curves, whereas the sticky price Phillips curve involves current and future expected inflation) precludes past or future expected inflation targeting to act as substitute for monetary policy's reaction to current inflation. Our study complements Holden's (2024) proposed real rate rules that follow the Taylor principle and incorporate a model specific real rate as part of the systematic component of the interest rate rule to ensure determinacy, but we show that his real rate addition is unnecessary and inconsequential in models that lack a long-run

[1] See Bullard and Mitra (2002) and Lubik and Marzo (2007) for compendia of determinacy results in sticky price models, extended by Loisel (2022) to a class of models with a wide of a lead-lag dependence. Benhabib, Schmitt-Grohé, and Uribe (2001) and Cochrane (2011) give critical views on (local) determinacy.

trade off between inflation and the real economy. That is, the Taylor principle is necessary in a much stricter sense than sticky price analyses imply, closing a gap in the literature by deriving the determinacy bounds in models with vertical long run Phillips curves.

Our baseline analysis examines three equation New Keynesian models where we replace the sticky price Phillips curve with two information based rigidity alternatives from the literature: sticky information and imperfect common knowledge. We connect these two models by their common vertical long run Phillips curves and add a third alternative that avoids taking a stand on the specifics of the rigidities facing supply and is specified solely by imposing the verticality of the supply relationship at some finite horizon directly. We prove that all three models have the same determinacy properties and that the restrictions so imposed on monetary policy's Taylor rule are more conservative than those imposed by the sticky price framework. After extending the analysis to more general Taylor rules, we move beyond the standard three equation framework and confirm the more conservative restrictions on monetary policy than under sticky prices in the regime switching environment of Davig and Leeper (2007) and with active/passive fiscal policy considerations, the fiscal theory, following Leeper (1991), including a numerical confirmation of the more conservative recommendations in the setting of Canzoneri, Cumby, and Diba (2010) and Canzoneri, Cumby, Diba, and López-Salido (2011) that imposes a long-run interdependence between the nominal and the real sides of the economy with their model of imperfectly substitutable, liquid government bonds.

In the sticky information model, Mankiw and Reis (2002) assume that firms update their information in an infrequent manner, i.e., firms adjust their prices delayed but optimally in response to their information sets. Mankiw and Reis (2007), Branch (2007) or Coibion and Gorodnichenko (2015a) and subsequent literature support the sticky information model as it outperforms the standard New Keynesian model and improves the dynamics of macroeconomic responses to monetary policy, precluding disinflationary expansions and attenuated responses to anticipated shocks and persistent zero lower bound episodes.[2] Central to the different role of monetary policy is the sticky information

[2]Further support comes from empirical evidence on the formation of macroeconomic expectations. Coibion and Gorodnichenko (2015a), Mertens and Nason (2020), Nason and Smith (2021), amongst others, show that stickiness in survey forecasts crucially depends on the inflation process. Andrade and Le Bihan (2013), Roth and Wohlfart (2020), Reis (2020), Cornand and Hubert (2022), Carroll, Crawley, Slacalek, Tokuoka, and White (2020), Link, Peichl, Roth, and Wohlfart (2023) document systematic biases in expectations and disagreement in inflation expectations among various types of agents tracing back to information rigidity.

model's vertical long-run Phillips curve, even out of equilibrium, whereas the sticky-price model imposes a systematic relationship between inflation and output, stable even in the long run, see, e.g., Woodford (2003, p. 254) or Galí (2008, p. 78).

Imperfect common knowledge among firms posits another source of price inertia. Contrary to the sticky information model suppliers in a common knowledge framework choose their prices based on their noisy observations. Woodford (2001a) demonstrates that both unanticipated and anticipated monetary shocks have real effects in the Lucas (1973) island model as higher order expectations - the expectations of others' expectations - persist in the absence of public information.[3] Adam (2007) analyzes optimal monetary policy in a model of nominal demand with imperfect common knowledge and flexible prices and Nimark (2008) combines Calvo price setting with noisy information and derives an imperfect common knowledge Phillips curve. Nimark's (2008) analysis in a general equilibrium model with New Keynesian IS demand and an interest rate rule is silent about determinacy and this is where our analysis picks up the standard. This model is a poignant alternative specifically as Angeletos and Lian (2018) demonstrate that this imperfect common knowledge approach rectifies the resolution of paradoxical prediction of New Keynesian models with Mankiw and Reis's (2002) sticky information by Chung, Herbst, and Kiley (2015) and Kiley (2016) with the micro data on price stickiness.[4]

Our analysis contributes to the literature on monetary policy in economies with limited information[5] that can provide markedly different policy recommendations than in full information settings like the canonical sticky price framework. Beginning with Ball,

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Chou, Easaw, and Minford (2023) estimate different models with incomplete information structures and show that Mankiw and Reis's (2002) sticky information generates a persistent and delayed response of inflation and output gap to a monetary policy shock empirically and An, Abo-Zaid, and Sheng (2023) estimate a sticky information model with endogenous inattention using US survey data and show that monetary policy's impact on the economy is amplified when both firms and households agents are inattentive.

[3]More recently, Acharya, Benhabib, and Huo (2021) and Huo and Takayama (2022) show that changes in agents' beliefs due to information frictions lead to persistent aggregate fluctuations.

[4]Angeletos and Huo (2021) address the endogeneity of information in dynamic beauty contests relate to Nimark's (2008) assumption of common knowledge of rationality and support their dynamic orthogonalization of innovation information by appealing to its nesting of Mankiw and Reis's (2002) sticky information as a special case. This close conceptual relationship is echoed by Chahrour and Jurado (2018) who provide an equivalence between news and noise in agents' beliefs and Coibion and Gorodnichenko (2015b) who find that sticky-information and noisy-information models both point to the same relationship between ex post mean forecast errors and ex ante mean forecast revisions.

[5]See Hellwig, Kohls, and Veldkamp (2012) for a unified framework.

Mankiw, and Reis (2005) who consider information stickiness in price setting which leads monetary policy to favor price level over inflation targeting. Angeletos, Iovino, and La'O (2016) show that incomplete information leads to nominal rigidities which can be neutralized by the conduct of monetary policy in the sticky price framework. Paciello and Wiederholt (2014) study optimal policy when firms are rationally inattentive to the state of the economy. Angeletos and La'O (2020) extend the "leaning against the wind" policy to firms' information-dependent actions. Bernstein and Kamdar (2023) and Iovino, La'O, and Mascarenhas (2022) examine the effects of informationally constrained policy makers. Ou, Zhang, and Zhang (2021) find that combining the Calvo friction with imperfect common knowledge leads to two price dispersion welfare channels associated with each of these frictions - while the channels separate individually in the absence of the other, their coexistence causes spillovers between the two rigidities. We use the standard Blanchard and Kahn (1980) concept of local determinacy - existence of a unique stationary, causal equilibrium - and while this is not the only concept,[6] our analysis provides the Taylor principle determinacy bounds on monetary policy thus far missing for economies with limited information but is silent as to these alternate equilibrium concepts. To enable the analysis, we derive recursive representations of the Phillips curves, in the frequency domain for sticky information and in a higher order expectations operator for imperfect common knowledge, enabling both novel interpretations and the analysis of determinacy in both models. These recursions allow us to separate the long run dynamic restrictions from sequences of prediction/forecast errors responsible for the rich shorter run dynamics, exactly analogously to our generic model of finite informational inertia.

2. Existence and Uniqueness: Frictionless and Sticky Prices

We fix ideas by first reviewing the conditions for determinacy in two basic models from the literature, a frictionless model of Cochrane (2011) and the textbook New Keynesian model, Woodford (2003) or Galí (2008). When there is separation between the nominal and real sides of the economy, output gap targeting is by construction irrelevant and the Taylor principle holds in a strict sense. This is no longer the case with the New Keynesian Phillips curve which allows monetary policy to substitute a concern for the output gap for its concern for inflation. The remainder of the paper will argue that the former results

[6] For example, appealing to coordination concepts of uniqueness, Angeletos and Lian (2023) reformulate the New Keynesian model as a dynamic game under imperfect information and Acharya, Benhabib, and Huo (2021) examine sentiment equilibria in beauty game framework.

hold more generally in models of informational inertia and that the latter is dubious as it rests on the New Keynesian Phillips curve failing to satisfy the natural rate hypothesis due to it remaining non vertical even in the long run.

We will address determinacy in the main analysis with three equation models: an interest rate rule and a demand and supply equation. We begin with the interest rate rule

$$R_t = \phi_\pi \pi_t + \phi_y y_t \tag{1}$$

where R_t is the nominal interest rate, π_t inflation, and y_t is the output gap. We will assume nonnegative coefficients, $\phi_\pi, \phi_y \geq 0$, unless otherwise noted. Demand will be described by a standard dynamic IS equation

$$y_t = E_t y_{t+1} - \sigma R_t + \sigma E_t \pi_{t+1} \tag{2}$$

where σ is the relative risk aversion or the inverse elasticity of intertemporal substitution. In our analysis, the differences between models will be entirely in the supply side.

In a frictionless model, the output gap is closed by definition and supply is

$$y_t = 0 \tag{3}$$

Note that this corresponds to the Cochrane's (2011) simplest model to address determinacy, also used by Davig and Leeper (2007), Lubik, Matthes, and Mertens (2023), and Holden (2024): the Fisher equation in a frictionless setting. The loglinear Fisher equation is

$$R_t = rr_t + E_t \pi_{t+1} \tag{4}$$

where rr_t the real rate. In the frictionless setting, the output gap is closed, $y_t = 0$ and the real rate is determined exogenously, e.g., by expected productivity growth - we set it to zero without loss of generality[7] and this corresponds to (2) with (3). We summarize determinacy in the following theorem

[7]This is the key difference to Holden (2024) who focuses on the leverage that real rates offer monetary policy in many popular macroeconomic models. We show that this approach is misguided, as the real rate will only be endogenous in the long run and, hence, offer monetary policy leverage for determinacy if the model in question posits a long run tradeoff between inflation and the real side of economy. When such a tradeoff is not present, as we argue with multiple models in this paper, Holden's (2024) real rate component is irrelevant and the original Taylor principle is a "robust rule."

Theorem 1 (Frictionless Determinacy). [8] *The frictionless model, given by (2), (3), with the Taylor rule (1), has a unique, stable equilibrium if and only if*

$$\phi_\pi > 1 \tag{5}$$

Proof. Combining (1), (2), and (3) gives

$$\phi_\pi \pi_t = E_t \pi_{t+1} \tag{6}$$

Solving forward following Blanchard (1979) gives $\pi_t = \lim_{j\to\infty} \frac{1}{\phi_\pi^j} E_t \pi_{t+j}$ which delivers a unique, bounded solution for π_t if and only if $1 < \phi_\pi$. □

Hence, the nominal interest rate must move more than one for one with inflation for the equilibrium to be determinate, giving us the celebrated Taylor principle.[9] Importantly, the degree of output gap targeting ϕ_y is irrelevant for determinacy, as the output gap is always zero here and offers no leverage in fulfilling the Taylor principle.

Determinacy is dramatically different when we replace the frictionless supply side (3) with the standard linear New Keynesian sticky price Phillips curve (NKPC) with Calvo (1983) contracts, e.g., Woodford (2003, p. 246) or Galí (2008, p. 49), given by

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t \tag{8}$$

where $0 < \beta < 1$ is the representative household's idiosyncratic discount factor and $\kappa = \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta$ is the slope of the short run Phillips curve with $0 \le 1-\theta \le 1$ being the probability of a price update - that is, θ is the Calvo (1983) sticky price parameter that measures the degree of nominal rigidity - and Θ collects other parameters, such as the link

[8] Unless otherwise stated, like here, proofs are in the appendix.

[9] We abstract from exogenous shocks in our analysis - this is without loss of generality, see e.g., Theorem 3.15 of Elaydi (2005) - the solution to a system of difference equations can be split into a particular and a homogenous solution and only the homogenous solution is relevant for the examination of determinacy. Following Taylor (1986), the bounded solution will be unique for any given bounded exogenous sequence of shocks iff the homogenous solution is uniquely determined by the boundedness conditions on the endogenous variables. Analogous conclusions can be found in Woodford (2003, pp. 252, & 636) and this follows the analysis of Lubik and Marzo (2007) for the sticky price model that follows. To see this consider the frictionless setup of Davig and Leeper (2007) with $rr_t = \rho rr_{t-1} + \epsilon_t$, $|\rho| < 1$, and ϵ_t is iid mean zero. The equations (4) and (1) imply $\phi_\pi \pi_t = E_t \pi_{t+1} + rr_t$. Solving forward, Blanchard (1979)

$$\pi_t = \lim_{j\to\infty} \frac{1}{\phi_\pi^j} E_t \pi_{t+j} + \sum_{j=0}^{\infty} \frac{1}{\phi_\pi^{j+1}} E_t rr_{t+j} = \lim_{j\to\infty} \frac{1}{\phi_\pi^j} E_t \pi_{t+j} + \frac{1}{\phi_\pi} \sum_{j=0}^{\infty} \left(\frac{\rho}{\phi_\pi}\right)^j rr_t \tag{7}$$

which *again* delivers a unique, bounded solution for π_t if and only if $1 < \phi_\pi$. This solution is $\pi_t = \frac{1}{\phi_\pi - \rho} v_t$ which is the particular solution $\frac{1}{\phi_\pi - \rho} v_t$ plus our homogenous solution $\pi_t = 0$ in theorem 1.

between marginal costs and the output gap, the interaction between returns to scale, etc. Hence, inflation today is a function of current output gap and future expected inflation.

Theorem 2 (Sticky Price Determinacy)**.** *The sticky-price model, given by (2), (8), with the Taylor rule (1), has a unique, stable equilibrium if and only if*

$$1-\frac{1-\beta}{\kappa}\phi_y<\phi_\pi \tag{9}$$

Proof. See the appendix and cf., Woodford (2003), Galí (2008), Bullard and Mitra (2002), or Lubik and Marzo (2007). We include it here to make analyzing determinacy in the information inertia models more accessible. □

Given the Taylor rule (1), the monetary authority can target inflation as well as the output gap to stabilize the economy - Woodford (2003, pp. 254–255), "... indeed, a large enough [response to] *either* [the output gap or inflation] suffices to guarantee determinacy". Indeed, the real rate can be raised in response to an off equilibrium inflation increase even by responding to output movements alone. Notice that this possibility disappears if $\beta = 1$ - however this is misleading as although an *average* long-run tradeoff disappears in this case, a dynamic one remains $\frac{\pi_t - E_t\pi_{t+1}}{\kappa} = y_t$ which monetary policy needs for its targeting of inflation (or output) at different horizons to translate into a response to current inflation as we will see later in our analysis of extended Taylor rules.

Consider as an intuitive alternative the expectational Phillips curve of Lucas (1973), expressed in terms of inflation and abstracting from shocks

$$y_t=\alpha\left(\pi_t-E_{t-1}\pi_t\right) \tag{10}$$

where $\alpha \geq 0$ is the (short run) slope of the Phillips curve that predicts output gaps from unexpected inflation, i.e. forecast errors.

Theorem 3 (Lucas (1973) Determinacy)**.** *The expectational Phillips curve model, given by (2), (10), with the Taylor rule (1), has a unique, stable equilibrium if and only if*

$$\phi_\pi>1 \tag{11}$$

Notice now that despite the fact that we are using a different supply side, the expectational Phillips curve of Lucas (1973), we have the **same** determinacy bounds on monetary policy as in the frictionless case of theorem 1.[10] Both of these models, in contrast to the

[10]Note that $\left(1+\sigma\phi_y\right)\alpha+\sigma\phi_\pi \neq 0$ if we are to be able to recover π_t from $E_{t-1}\pi_t$. As we only consider $\phi_\pi, \phi_y \geq 0$, this holds here with certainty. As we shall see later, for more general models, this might not

sticky price model, have vertical long run Phillips curves. Specifically, the frictionless model has a vertical Phillips curve at every horizon - from (3) $y_t = 0$ - and the Lucas (1973) supply side at the one period horizon. To see this, take the $t-1$ expectation of (10)

$$E_{t-1}y_t = \alpha E_{t-1}[\pi_t - E_{t-1}\pi_t] = 0 \tag{14}$$

The slope of the long-run Phillips curve following King and Watson (1994) is

Definition 1 (Long-Run Phillips Curve Slope)**.** *The inverse of the long-run Phillips curve slope, LRS, or the long-run trade-off between the output gap and inflation is given by*

$$1/LRS \equiv \lim_{k\to\infty} \frac{\partial y_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} \tag{15}$$

following King and Watson (1994) where ϵ_t is a demand shock.

We can append the IS equation (2) with just such a demand shock to make the connection of our analysis to this definition explicit

$$y_t = E_t y_{t+1} - \sigma R_t + \sigma E_t \pi_{t+1} + \epsilon_t \tag{16}$$

but the slope of the Phillips curve follows from the properties of the Phillips curve and we could just as well consider an expectations or sunspot shock. In either case, stationary causal representations of y_t and π_t with respect to ϵ_t can be written as

$$y_t = \sum_{k=0}^{\infty} y_k \epsilon_{t-k} + \text{t.i.o.}\epsilon, \qquad \pi_t = \sum_{k=0}^{\infty} \pi_k \epsilon_{t-k} + \text{t.i.o.}\epsilon \tag{17}$$

where "t.i.o.ϵ" are terms independent of ϵ and $x_k, x = y, \pi$ are the MA coefficients with respect to ϵ: $x_k = \frac{\partial x_t}{\partial \epsilon_{t-k}} = \frac{\partial x_{t+k}}{\partial \epsilon_t}$.

For the frictionless model, the output gap is always closed, $y_t = 0$ from (3), so $y_k = 0 \forall k$ or $\frac{\partial y_{t+k}}{\partial \epsilon_t} = 0$ and the Phillips curve is vertical at every horizon and including the long run

$$1/LRS|_{Frictionless} = \lim_{k\to\infty} \frac{\partial y_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} = \lim_{k\to\infty} 0 \left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} = 0 \tag{18}$$

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hold everywhere, but is simply a singularity in the parameter space that prevents the unique resolution of the prediction error(s). Additionally, in the frictionless model, the output gap was always closed and hence output was determined apart from monetary policy: an indeterminacy would only be a nominal indeterminacy that afflicted π_t and, hence, R_t. This is not true here as an indeterminacy in $E_{t-1}\pi_t$ would lead to an indeterminacy in y_t through the Phillips curve (10)

$$y_t = \alpha(\pi_t - E_{t-1}\pi_t) = \frac{\alpha}{(1+\sigma\phi_y)\alpha + \sigma\phi_\pi}\left(\sigma E_t\pi_{t+1} + (1+\sigma\phi_y)\alpha E_{t-1}\pi_t\right) - \alpha E_{t-1}\pi_t \tag{12}$$

$$= \frac{\alpha\sigma}{(1+\sigma\phi_y)\alpha + \sigma\phi_\pi}\left(E_t\pi_{t+1} - \phi_\pi E_{t-1}\pi_t\right) \tag{13}$$

In Lucas's (1973) Phillips curve, inserting (17) and matching terms on ϵ_{t-j} gives

$$y_k = \begin{cases} \alpha\pi_k & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

so as $\frac{\partial y_{t+k}}{\partial \epsilon_t} = 0, \forall k > 0$ the Phillips curve is vertical at every horizon beyond the period of impact of a shock and hence it is also vertical in the long run

$$1/LRS|_{Lucas} = \lim_{k\to\infty} \frac{\partial y_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} = \lim_{k\to\infty} 0 \left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} = 0 \tag{20}$$

Thus both the frictionless and Lucas's (1973) expectational Phillips curve share a common long run Phillips curve, namely a vertical one.

For the sticky price Phillips curve (8), inserting (17) and matching terms on ϵ_{t-j} gives

$$\pi_k = \beta\pi_{k+1} + \kappa y_k \tag{21}$$

solving for y_k and inserting into (15) in definition 1

$$\frac{\partial y_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} = \frac{1}{\kappa}\left(1 - \beta\frac{\pi_{k+1}}{\pi_k}\right) \tag{22}$$

and the long run Phillips curve is not vertical but instead

$$1/LRS|_{StickyPrice} = \lim_{k\to\infty} \frac{\partial y_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} = \lim_{k\to\infty} \frac{1}{\kappa}\left(1 - \beta\frac{\pi_{k+1}}{\pi_k}\right) = \frac{1}{\kappa}\left(1 - \beta\lim_{k\to\infty}\frac{\pi_{k+1}}{\pi_k}\right) \tag{23}$$

If we consider the threshold of a permanent, constant effect on inflation, $\lim_{k\to\infty}\pi_{k+1} = \lim_{k\to\infty}\pi_k$ such that $\lim_{k\to\infty}\frac{\pi_{k+1}}{\pi_k} = 1$, this becomes

$$1/LRS|_{StickyPrice, \lim_{k\to\infty}\frac{\pi_{k+1}}{\pi_k}=1} = \frac{1-\beta}{\kappa} \tag{24}$$

which is readily identifiable as the rate with which a response to the output gap, ϕ_y, could be substituted for a direct response to inflation, ϕ_π, in the determinacy condition (A.5) for the sticky price model. That is, it is the non vertical Phillips curve in the long run that enables monetary policy to substitute a response to the output gap for a response to inflation and the rate at which it can substitute is the slope of the long run Phillips curve.

The equivalence between the determinacy bounds on the frictionless model and with Lucas's (1973) Phillips curve is not coincidental and we will now show that this equivalence holds more generally with Phillips curves that become vertical in the long run. We will begin by introducing three different Phillips curves and how they relate to a long run vertical curve before we then turn to their determinacy and confirm the conditions are identical to the frictionless and Lucas's (1973) expectational Phillips curve models here.

3. Phillips Curves of Imperfect Information

In this section, we review two Phillips curves - sticky information and imperfect common knowledge - and introduce a general finite informational inertia supply side. The first two are examples of Angeletos and Lian's (2018) "two leading forms of learning" where agents either become gradually aware of or receive private signals about fundamentals and the third is formulated without appealing to any specific rigidity, form of learning, etc., but rather only to deliver a vertical Phillips curve at an arbitrary horizon. We juxtapose these three alongside the sticky price Phillips curve from (8) and relate them to the natural rate hypothesis (NRH), a concept inextricably related to the verticality of the Phillips curve.

3.1. Sticky Information Phillips Curve - Recursive in the Frequency Domain

The sticky information Phillips curve has an infinite regress of price plans or lagged expectations that cannot be expressed recursively in the time domain, precluding the application of standard DSGE techniques to assess determinacy. We prove in the following, however, that the sticky information Phillips curve does have a recursive representation in the frequency domain, which will enable our analytic determinacy analysis.[11] To make the analysis approachable, we review first the canonical sticky price and present its frequency domain before we turn to the sticky information model. The frequency domain provides a novel, fundamental perspective on the sticky information Phillips curve, while merely providing an alternative representation for the sticky price Phillips curve.

We will have determinacy if there is a unique bounded or stable solution, c.f. Blanchard and Kahn (1980) and Klein (2000), which are the unique, causal Wold representations of the endogenous variables as mean zero, linearly regular covariance stationary stochastic process from Whiteman's (1983), i.e., for each endogenous variable x_t, a unique sequence $\left\{x_j^\epsilon\right\}_{j\geq 0}$ with $x_j^\epsilon = 0, \forall j < 0$ in

$$x_t^\epsilon = x^\epsilon(L)\epsilon_t = \sum_{j=0}^{\infty} x_j^\epsilon \epsilon_{t-j} \tag{25}$$

with $\sum_{j=0}^{\infty} x_j^{\epsilon 2} < \infty$ and L the lag operator $Lx_t = x_{t-1}$ and ϵ_t is a mean zero fundamental innovation.[12] Again, we are concerned with the uniqueness of equilibria and not their values, the actual source of this shock is immaterial - in the linear framework it will alter

[11] We do not include shocks explicitly, see also footnote 9, and we are defining the processes in terms of the kernel of the operator that defines the linear rational expectations model, see Al-Sadoon (2024).

[12] Note that we are abusing notation somewhat and choosing to use the same letter x to refer to a discrete time series, x_t, as well as that variable's transform function $x(z)$ or MA representation/response

the particular/inhomogenous solution but will leave the homogenous solution unchanged. Hence we will drop the ϵ superscript.[13] Using the z-transform, $x(z)$, of the sequence $\left\{x_j^{\epsilon}\right\}_{j\geq 0}$, $x(z)=\sum_{j=0}^{\infty} x_j z^j$, we can use the Wiener-Kolmogorov prediction formula to yield $E_t[x_{t+n}] = E_t\left[\sum_{j=0}^{\infty} x_j \epsilon_{t-j+n}\right] = \sum_{j=0}^{\infty} x_{j+n}\epsilon_{t-j}$. The Wiener-Kolmogorov prediction formula of "plussing" gives the frequency domain version

$$\mathcal{Z}\{E_t[x_{t+1}]\} = \left[\frac{x(z)}{z}\right]_+ = \frac{1}{z}(x(z)-x(0)) \tag{26}$$

where $_+$ is the annihilation operator, see Sargent (1987a) and Hamilton (1994). Following the Riesz-Fisher theorem, see Sargent (1987a), Whiteman and Lewis (2018), and Tan and Walker (2015) alongside Rudin (1987, pp. 85-92), the requirement that x_t be causal with $\sum_{j=0}^{\infty} x_j{}^2 < \infty$ is equivalent to $x(z)$ being a squared-integrable analytic function on the unit disk, a function from a Hardy space, $x(z) \in H^2$. Defining $z = Re^{-i\omega}$, where R is the radius and ω the angular frequency, we get $x(e^{-i\omega})$ for $R=1$ as transfer function from which we can, e.g., calculate the spectrum and recover covariances using inverse Fourier transforms, see Sargent (1987a) and Hamilton (1994). Note finally, see the online appendix for more details, that from $x(z)=\sum_{j=0}^{\infty} x_j z^j$ we obtain the impact response of x_t to a shock ϵ_t from evaluating $x(z)$ at $z=0$ as $x(0)=x_0$ and, additionally, the cumulative response from evaluating $x(z)$ at $z=0$: $x(1)=\sum_{j=0}^{\infty} x_j$.

We can apply the z-transform to our equilibrium conditions to use this representation and we begin with the familiar sticky price NKPC (8), $\pi_t = \beta E_t \pi_{t+1} + \kappa y_t$ which gives

$$\pi(z) = \beta\frac{1}{z}(\pi(z)-\pi_0) + \kappa y(z) \tag{27}$$

This implies that inflation and the output gap are linked at all z and determinacy boils down to being able to pin down the impact responses such as π_0 uniquely.

This is not true, however, for all models including the sticky information Phillips curve, which implements probabilistic contracts of predetermined prices in the vein of Fischer (1977) with the Calvo (1983) mechanism.[14] Mankiw and Reis's (2002) approach yields the

to a fundamental process j periods ago, x_j. This serves to save on the verbosity of notation, which might otherwise read $x_t = \sum_{j=0}^{\infty} \delta_j^x \epsilon_{t-j}$ following, e.g., Meyer-Gohde (2010).

[13]In other words, we assess the uniqueness of the zero solution of the endogenous variables in the absence of fundamental shocks. Thus ϵ_t can be interpreted as a sunspot or nonfundamental shock and we are ascertaining the uniqueness of the zero solution - i.e., no response to sunspots.

[14]See Bénassy (2002, Ch. 10), Mankiw and Reis (2002), and Devereux and Yetman (2003).

following aggregate supply equation

$$\pi_t = \frac{1-\lambda}{\lambda}\xi y_t + (1-\lambda)\sum_{i=0}^{\infty}\lambda^i E_{t-i-1}[\pi_t + \xi(y_t - y_{t-1})] \tag{28}$$

where y_t is the output gap, π_t inflation, $\xi > 0$ measures the degree of strategic complementarities, and $0 < 1-\lambda < 1$ is the probability of an information update. The infinite regress of lagged expectations precludes a recursive representation in the time domain.

Lagged expectations ($E_{t-i}[x_t],\ i > 0$), Whiteman's (1983) "withholding equations", can be represented in the z domain via the Wiener-Kolmogorov prediction formula (26) as

$$\mathcal{Z}\{E_{t-i}[x_t]\} = z^i\left[\frac{x(z)}{z^i}\right]_+ = x(z) - \sum_{j=0}^{i} x^j(0)z^j \tag{29}$$

where $x^j(0)$ is the $j'th$ derivative of $x(z)$ evaluated at the origin, or x_j from the sequence for Wold representation $x(z) = \sum_{j=0}^{\infty} x_j z^j$. These withholding equations by themselves are not sufficient to solve models like those involving the sticky information Phillips curve (28), as it requires an *infinite* number of withholding equations.[15] Using (29), the sticky information Phillips curve (28) can be expressed as

$$\pi(z) = \frac{1-\lambda}{\lambda}\xi y(z) + (1-\lambda)\sum_{i=0}^{\infty}\lambda^i\left[\pi(z) - \sum_{j=0}^{i}\pi^j(0)z^j + \xi(1-z)\left(y(z) - \sum_{j=0}^{i} y^j(0)z^j\right)\right] \tag{30}$$

The infinite sums in (30) can be resolved by noting that:[16]

$$\sum_{i=0}^{\infty}\lambda^i\left[x(z) - \sum_{j=0}^{i} x_j z^j\right] = \frac{1}{1-\lambda}x(z) - \sum_{i=0}^{\infty}\lambda^i\sum_{j=0}^{i} x_j z^j = \frac{1}{1-\lambda}x(z) - \sum_{j=0}^{\infty}\sum_{i=j}^{\infty} x_j z^j \lambda^i \tag{31}$$

$$= \frac{1}{1-\lambda}x(z) - \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\lambda^i x_j z^j \lambda^j = \frac{1}{1-\lambda}x(z) - \sum_{j=0}^{\infty}\frac{1}{1-\lambda}\lambda^i x_j z^j \lambda^j \tag{32}$$

$$= \frac{1}{1-\lambda}(x(z) - x(\lambda z)) \tag{33}$$

Combining these results we get the following representation of the Phillips curve (28)

$$\pi(z) = \xi\left(\frac{1-\lambda}{\lambda}\right)y(z) + \pi(z) - \pi(\lambda z) + \xi(1-z)y(z) - \xi(1-\lambda z)y(\lambda z) \tag{34}$$

[15] Tan and Walker (2015, p. 99) claim that their method can be "easily adapted" to the sticky information model using withholding equations by "replacing E_t with E_{t-j} for any finite j." This is misleading or incomplete, as the sticky information model involves lagged information that reaches back past any finite j.

[16] The exchange of the order of summation follows from our assumption of time-independent square-summable linear processes. See also the online appendix.

collecting terms gives $\xi(1-\lambda z)y(z) = \lambda\pi(z\lambda) + \xi\lambda(1-\lambda z)y(\lambda z)$ which we rearrange to yield the following representation of the sticky information Phillips curve

$$\xi\left(\frac{1}{\lambda} - z\right)y(z) = \pi(\lambda z) + \xi(1-\lambda z)y(\lambda z) \tag{35}$$

For any point $z = Re^{-i\omega}$ on the unit disk, the output gap $y(z)$ is driven by inflation and by the output gap evaluated at the corresponding radially contracted point $\lambda z = (\lambda R)e^{-i\omega}$. The factor $\lambda R < R$ pulls z radially toward the origin, geometrically damping every lag coefficient by λ^j. As our functions $y(z)$ and $\pi(z)$ are analytic on the unit disk, interior points (say $y(\lambda e^{-i\omega})$) can be recovered from the boundary function, in our case the unit circle (say $y(e^{-i\omega})$), via a Poisson integral, a standard result that can be found in, e.g., Theorem 11.8 in Rudin (1987). This integral smooths using the Poisson kernel that implements a low pass or averaging smoothing from the frequency perspective as we summarize in the following proposition [17]

Proposition 1 (Radial contracting, Poisson smoothing, memory, and the spectrum)**.** *For a mean zero, linearly regular covariance stationary stochastic process x_t with autocovariance $\gamma(m)$ and the causal transfer function $x(z)$ in H^2*

$$x_t = x(L)\varepsilon_t = \sum_{j=0}^{\infty} x_j\varepsilon_{t-j}, \quad \sum_{j=0}^{\infty} x_j^2 < \infty \tag{36}$$

where ε_t is white noise with variance σ^2

(1) For $0 < \lambda < 1$ and all $\omega \in [-\pi,\pi]$, the λ damped $x(\lambda z)$ can be recovered from the original $x(z)$ via the Poison integral or convolution with the Poisson kernel

$$x(\lambda e^{i\omega}) = (P_\lambda * x)(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_\lambda(\omega-\theta)x(e^{i\theta})d\theta \tag{37}$$

where

$$P_\lambda(\phi) = \frac{1-\lambda^2}{\left|1-\lambda e^{-i\omega}\right|^2} = \frac{1-\lambda^2}{1-2\lambda\cos\phi+\lambda^2} \tag{38}$$

(2) Convolution with the Poisson kernel gives $x(\lambda e^{i\omega})$ as a low–pass average of the values of $x(z)$ on the unit circle, damping higher harmonics by multiplying the nth Fourier harmonic $e^{in\theta}$ by $\lambda^{|n|}$ and, equivalently, damping the impulse response at lag j by λ^j via geometric decay

[17] We provide intuitive applications to ARMA processes in the online appendix and for a simple intuition for this contractionary effect, realize that a $|\rho| < 1$ converts a random walk transfer function $x_{RW}(z) = 1/(1-z)$ into an AR(1), $x_{RW}(\rho z) = 1/(1-\rho z) = x_{AR(1)}(z)$ with an autocorrelation of ρ, see also Priestley (1981, pp. 238-9) and Oppenheim and Schafer (2010, Section 5.3.1).

(3) *Radial contraction by λ shortens memory by at least an exponential factor $\lambda^{|m|}$*
(4) *The spectrum at radius λ is $f_x^{(\lambda)}(\omega) = \frac{\sigma^2}{2\pi}|x(\lambda e^{-i\omega})|^2$. For an invertible, fundamental process x_t, $\log f_x^{(\lambda)} = (P_\lambda * \log f_x)$ with $f_x(\omega) = \frac{\sigma^2}{2\pi}|x(e^{-i\omega})|^2$ and λ induces geometric-mean flattening with $\log f_x^{(\lambda)} \to \frac{1}{2\pi}\int_{-\pi}^{\pi} \log f_x$ and $x(\lambda z)$ to white noise as $\lambda \to 0$*

The Poisson integral or Poisson kernel smoothing convolution in (37) functions as a low pass filter with λ as the damping or bandwidth parameter, see especially Li (1997) for time series interpretations. As the Poisson kernel $P_\lambda(\phi)$ in (38) is even, positive, and integrates to one, (Rudin, 1987, p. 233) and (Duren, 1970, section 1.1) this smoothing preserves the total power or energy of the spectrum. So as λ decreases from 1, the kernel smoothes out higher frequency contributions from $f(\theta)$, hence pushing the power towards lower frequencies. As $\lambda \to 0$, $P_\lambda(\phi)$ smoothes the spectrum towards white noise, compressing the spectral energy towards the origin.

From (35), the output gap at a given point, $z = Re^{-i\omega}$, depends on inflation and itself at damped point, λz. With a high updating probability, or λ small, the geometric weight λ^j damps distant lags heavily, so only recent inflation influences y. With a low updating probability, or λ near to 1, the damping is weak: higher-order lags persist and low-frequency inflation has a larger impact on the output gap. We can rearrange (35) through recursive substitution in $\lambda^j z$ to express this more clearly as a recursion

$$y(z) = \frac{\lambda}{\xi}\frac{1}{1-\lambda z}\pi(\lambda z) + \lambda y(\lambda z) \tag{39}$$

This gives the output gap as a cascade of ever-damped components of inflation.

$$y(z) = \frac{1}{\xi}\sum_{j=1}^{\infty}\frac{\lambda^j}{1-\lambda^j z}\pi(\lambda^j z) \tag{40}$$

If the fraction of firms which get an information update, $1-\lambda$, is low (high) and hence λ closer to one (zero), the output gap is driven more (less) strongly by inflation at higher lags, that is $\tilde{R} = \lambda R$ is closer to R (zero). However, as long as $\lambda < 1$, there is no tradeoff between output gap and inflation in the long run in the sense that $y(z)$ and $\pi(z)$ are not connected on the unit circle. We derive this formally in the following[18]

Proposition 2 (Long run independence of the output gap from inflation under the sticky information Phillips curve)**.** *Given the recursive representation of the sticky information Phillips curve (39) and assume $0 < \lambda < 1$ and $\xi \neq 0$. Defining the operators $(C_\lambda f)(z) = f(\lambda z)$, $(M_G f)(z) = G(z)f(z)$ and $G(z) = \frac{\lambda/\xi}{1-\lambda z}$, then*

[18]We are grateful to an anonymous referee for suggesting this approach.

(1) There is no bounded linear operator B as $By = \pi$ that maps to H^2 for all $\pi \in H^2$

(2) equivalently, the sticky information Phillips curve map defined as $N = (I - \lambda C_\lambda)^{-1} M_G C_\lambda$ has no bounded left inverse.

Hence, $\pi \in H^2$ implies $y \in H^2$, but $y \in H^2$ does not imply $\pi \in H^2$.

It is the recursivity in the frequency domain that connects $y(z)$ to only $\pi(\lambda^k z)$ for positive k meaning that the output gap can be determined by inflation via the sticky information Phillips curve but not vice versa. It is the non-invertibility of this composition, C_λ, that prevents the outer structure and us from writing $B(z)y(z) = \pi(z)$ for a bounded B on the disk. This absence of a stable trade off between inflation and the output gap on the unit circle can be seen intuitively by noticing that assuming $\pi(z)$ is an analytic function on the unit disk implies that $y(z)$ is too, but the converse fails. That is, the Phillips curve determines the output gap from inflation and not vice-versa.

Accordingly, take π_t as a given mean zero, linearly regular covariance stationary stochastic process with known Wold representation, i.e., $\pi(z)$ as an analytic function with a region of convergence of at least $|z| \le 1$. Thus, $\pi(\lambda^j z)$ has a region of convergence of at least $|\lambda^j z| \le 1$, which as $0 < \lambda < 1$ is $|z| \le \lambda^{-j}$ and hence $\pi(\lambda^j z)$ has a region of convergence of at least $|z| \le 1$. Defining $\tilde{\pi}(\lambda^j z) \equiv \frac{1}{1-\lambda^j z}\pi(\lambda^j z)$, this will also have a region of convergence of at least $|z| \le 1$ for $0 < \lambda < 1$ as the pole $z \in \mathscr{C} : 1 - \lambda^j z = 0$ is outside the unit circle and the sum is convergent from the λ^j weights. Hence $y(z)$ over the unit disk is given by

$$y(z) = \frac{1}{\xi}\sum_{j=1}^{\infty} \lambda^j \tilde{\pi}(\lambda^j z) = \frac{1}{\xi}\sum_{j=1}^{\infty} \frac{\lambda^j}{1-\lambda^j z}\pi(\lambda^j z) \tag{41}$$

given $\pi(z)$ analytic over the unit disk. For the converse, take $y(z)$ as a given mean zero, linearly regular covariance stationary stochastic process with known Wold representation, an analytic function with a region of convergence of at least $|z| \le 1$. Starting from (39)

$$\pi(\lambda z) = \frac{\xi}{\lambda}(1-\lambda z)(y(z) - \lambda y(\lambda z)) \Rightarrow \pi(z) = \frac{\xi}{\lambda}(1-z)(y(z/\lambda) - \lambda y(z)) \tag{42}$$

Notice that $\pi(z)$ from this would demand that $y(z/\lambda)$ be analytic with a region of convergence of at least the unit disk. That is, $y(z)$ would need a region of convergence of at least $|z\lambda| \le 1$ or of at least $|z| \le 1/\lambda$ for $0 < \lambda < 1$, which is outside the unit circle. Thus, knowing $y(z)$ over the unit disk is insufficient to determine $\pi(z)$. That is, the sticky information Phillips curve determines the output gap from inflation and not the other way around.

Contrast this with the sticky price Phillips curve (27) rewritten as

$$\pi(z) = \frac{1}{1-\beta/z}\left(\kappa y(z) - \beta/z\pi(0)\right) \Rightarrow y(z) = \frac{1-\beta/z}{\kappa}\pi(z) + \frac{\beta}{\kappa}\frac{1}{z}\pi(0) \tag{43}$$

From (43) it follows directly that assuming π_t is a given mean zero, linearly regular covariance stationary stochastic process with known Wold representation, i.e., $\pi(z)$ as an analytic function with a region of convergence of at least $|z| \leq 1$, that the same holds for $y(z)$. For the converse, notice that as $0 < \beta < 1$ there is a pole $z \in \mathscr{C} : 1 - \beta/z = 0$ inside the unit circle, but the singularity at the pole $z = \beta$ can be removed via

$$\lim_{z \to \beta} \left(1 - \beta/z\right) \pi(z) \stackrel{!}{=} 0 = \kappa y(\beta) - \pi(0) \tag{44}$$

Thus, given a mean zero, linearly regular covariance stationary stochastic process with known Wold representation for $y(z)$, $\pi(z)$ is also an analytic function with a region of convergence of at least $|z| \leq 1$.[19] Hence, in contrast to sticky information, the sticky price Phillips curve *does* imply a stable long run tradeoff between inflation and the output gap.

This difference manifests itself in the slope of the long run Phillips curve, which becomes vertical in the long run, as pointed out in the time domain by Mankiw and Reis (2002), and we now confirm. Thus the sticky information model shares the same long run vertical Phillips curve with the frictionless and Lucas's (1973) expectational Phillips curves from above, which contrasts with the sticky price Phillips curve.

Proposition 3 (Long Run Phillips Curve Slope under Sticky Information)**.** *The (inverse) slope of the Phillips curve* $\frac{\partial y_{t+k}}{\partial \epsilon_t} \left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}$ *for the sticky information Phillips curve (28) is*

$$\frac{\lambda^{k+1}}{1-\lambda^{k+1}} \frac{1}{\xi} \left(\sum_{j=0}^{k} \frac{\partial \pi_{t+j}}{\partial \epsilon_t} \right) \left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t} \right]^{-1} \tag{45}$$

which in the long run, as $k \to \infty$ is

$$1/LRS|_{Sticky\ Information} = \lim_{k \to \infty} \left(\frac{\lambda^{k+1}}{1-\lambda^{k+1}} \frac{1}{\xi} \left(\sum_{j=0}^{k} \frac{\partial \pi_{t+j}}{\partial \epsilon_t} \right) \right) \left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t} \right]^{-1} = 0 \tag{46}$$

We find as above in the frictionless model and Lucas's (1973) supply side that the Phillips curve in the sticky information model is vertical in the long run. However in contrast to these two models, it is non vertical at every finite horizon.

3.2. Phillips Curves under Imperfect Common Knowledge

An alternative informational rigidity in the literature assumes imperfect common information. With firms having only incomplete, noisy signals on the state of the economy, be this assumed exogenously like Nimark (2008) or as the result of a capacity constraint like Adam (2007), they have different information sets and disagree about the state

[19] See the online appendix for details on this complex analysis approach.

of the economy. Nimark (2008) shows that the assumption of common knowledge of rationality enables the Phillips curve to be expressed via an infinite cascade of higher order expectations in the otherwise standard time recursive relation between current marginal costs and current and future inflation.[20] We show that this cascade of higher order expectations and the recursivity of the average higher order expectations allows us to express this cascade recursively in the signal space, giving us a compact representation of the imperfect common information Phillips curve.

Nimark (2008) presents a Phillips curve that embeds the standard sticky price approach into this imperfect information setup as follows

$$\pi_t = (1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k\, mc_{t|t}^{(k)} + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k\,\pi_{t+1|t}^{(k+1)} \tag{47}$$

where mc_t are the aggregate marginal costs and the remainder of variable names and parameters are identical to the sticky price Phillips curve in (8). Imperfect knowledge is encompassed in the variables on the right-hand side of (47) where

$$x_{t|t}^{(0)} \equiv x_t \qquad x_{t|s}^{(1)} \equiv \int E\left[x_t|\mathscr{I}_s(j)\right]dj \qquad x_{t|s}^{(k)} \equiv \int E\left[x_{t|s}^{(k-1)}x_t\middle|\mathscr{I}_s(j)\right]dj \tag{48}$$

where $\mathscr{I}_s(j)$ is agent j's information set at time s. Hence the imperfect common knowledge Phillips curve in (47) contains the infinite cascade of higher order beliefs, Townsend's (1983) forecasting the forecasts of others. Inflation depends not only on marginal costs and future expected inflation, but the average (via the integral over agents) *imperfect* expectation of marginal costs and future expected inflation, the average *imperfect* expectation of these average *imperfect* expectations, etc.

We express this Phillips curve recursively by defining the operator H_s

$$H_s x_t \equiv \int E\left[x_t|\mathscr{I}_s(j)\right]dj \tag{49}$$

which we call the average higher order expectations operator and rewriting (47) as

$$\pi_t = (1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k H_t^k mc_t + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k H_t^k H_t\pi_{t+1} \tag{50}$$

$$= \frac{(1-\theta)\left(1-\beta\theta\right)}{1-(1-\theta)H_t}mc_t + \frac{\beta\theta}{1-(1-\theta)H_t}H_t\pi_{t+1} \tag{51}$$

which follows as $0<1-\theta<1$. Multiplying both sides with $1-(1-\theta)H_t$ gives

$$(1-(1-\theta)H_t)\pi_t = (1-\theta)\left(1-\beta\theta\right)mc_t + \beta\theta H_t\pi_{t+1} \tag{52}$$

[20] Angeletos and Lian (2018) and Angeletos and Huo (2021) address the potential for dynamic higher order expectations to inhibit this time recursive relation.

or

$$\underbrace{(1-H_t)\pi_t}_{\text{A forecast/prediction error}} + \theta H_t \underbrace{\left(\pi_t - \beta\pi_{t+1}\right)}_{\substack{\text{Standard sticky price dynamic} \\ \text{inflation trade off}}} = (1-\theta)\left(1-\beta\theta\right)mc_t \tag{53}$$

Defining the full-information conditional expectation $E_s[\cdot] \equiv E[\cdot|\mathscr{I}_s]$ where $\mathscr{I}_s$ is the information set containing all public and private information up to and including time t, hence the union of all atomistic agents' information sets $\mathscr{I}_s(j)$ and all variables not (fully) revealed by their information. As $\mathscr{I}_s(j) \subseteq \mathscr{I}_s$ - that is, agents' individual information sets are all nested in the full-information set, and the law of iterated expectations applies with $H_tE_t\pi_{t+1} = H_t\pi_{t+1}$: $H_tE_t\pi_{t+1} = \int E\left[E\left[\pi_{t+1}|\mathscr{I}_t\right]|\mathscr{I}_t(j)\right]dj = \int E\left[\pi_{t+1}|\mathscr{I}_t(j)\right]dj = H_t\pi_{t+1}$.[21] Hence the Phillips curve can be written as

$$(1-H_t)\pi_t - \theta(1-H_t)\left(\pi_t - \beta E_t\pi_{t+1}\right) + \theta\left(\pi_t - \beta E_t\pi_{t+1}\right) = (1-\theta)\left(1-\beta\theta\right)mc_t \tag{54}$$

or

$$\pi_t = \beta E_t\pi_{t+1} + \frac{(1-\theta)\left(1-\beta\theta\right)}{\theta}\Theta y_t - \frac{1}{\theta}\zeta_t \tag{55}$$

with marginal costs and the output gap related through Θ as above and where ζ_t is a prediction/forecast error given by

$$\zeta_t = (1-\theta)(1-H_t)\pi_t + \theta\beta(1-H_t)E_t\pi_{t+1} \tag{56}$$

$$= (1-\theta)\int\left(\pi_t - E\left[\pi_t|\mathscr{I}_t(j)\right]\right)dj + \theta\beta\int\left(E_t\pi_{t+1} - E\left[\pi_{t+1}|\mathscr{I}_t(j)\right]\right)dj \tag{57}$$

Comparing (55) with the sticky price Phillips curve in (8), we surmise that imperfect common knowledge introduces forecast/prediction errors that can affect the dynamic, but does not alter the fundamental long run dynamic trade off.

[21] Note that the nesting property of the information sets gives $E\left[E_t\pi_{t+1}|\mathscr{I}_t(j)\right] = E\left[\pi_{t+1}|\mathscr{I}_t(j)\right]$ from which $H_tE_t\pi_{t+1} = H_t\pi_{t+1}$ also follows. The full-information expectation $E_t\pi_{t+1}$ is the best possible forecast of π_{t+1} given the entire economy's information at time t and as each agent understands this property of E_t (i.e., Nimark's (2008) "common knowledge of rationality"), their expectation of $E_t\pi_{t+1}$ is simply their own forecast $E\left[\pi_{t+1}|\mathscr{I}_t(j)\right]$. The law of iterated expectations however does not generically hold: for the average expectation $E_tH_t\pi_{t+1} = E\left[\int E\left[\pi_{t+1}|\mathscr{I}_t(j)\right]dj|\mathscr{I}_t\right] \neq H_t\pi_{t+1}$ and $\neq E_t\pi_{t+1}$ as H_t conditions on a collection of non-nested information sets, reflecting the hallmark result under dispersed information that agents condition expectations on different information sets and must estimate others' beliefs, leading to higher-order uncertainty and strategic distortions.

Proposition 4 (Long Run Phillips Curve Slope under Imperfect Common Knowledge)**.** *The (inverse) slope of the Phillips curve* $\frac{\partial y_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}$ *for the imperfect common knowledge Phillips curve (55) is*

$$\frac{1}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\left[\theta\left(\frac{\partial \pi_{t+k}}{\partial \epsilon_t}-\beta\frac{\partial \pi_{t+k+1}}{\partial \epsilon_t}\right)+\frac{\partial \zeta_{t+k}}{\partial \epsilon_t}\right]\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} \tag{58}$$

which in the long run, as $k \to \infty$ is

$$1/LRS|_{\textit{Imperfect Common Knowledge}} = \frac{\theta}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\left[1-\beta\lim_{k\to\infty}\frac{\pi_{k+1}}{\pi_k}\right] \tag{59}$$

To see this, solve (55) for y_t,

$$y_t = \frac{\theta}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\left(\pi_t-\beta E_t\pi_{t+1}\right)+\frac{1}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\zeta_t \tag{60}$$

Inserting (17) - analogously $\zeta_t = \sum_{k=0}^{\infty}\zeta_k\epsilon_{t-k} + \text{t.i.o.}\epsilon$ - and matching terms on ϵ_{t-j}

$$y_k = \frac{\theta}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\left(\pi_k-\beta\pi_{k+1}\right)+\frac{1}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\zeta_k \tag{61}$$

inserting into (15) in definition 1

$$\frac{\partial y_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} = \frac{\theta}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\left(1-\beta\frac{\pi_{k+1}}{\pi_k}\right)+\frac{1}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\frac{\zeta_k}{\pi_k} \tag{62}$$

Nimark's (2008) assumption of common knowledge of rationality eliminates the higher order expectation errors that comprise ζ_t in response to a single shock over time as each agent's Bayesian updating process rectifies initial disagreement and precludes any permanent misalignment, delivering $\zeta_k \underset{k\to\infty}{\to} 0$. Yet his model also contains a Calvo sticky price friction and the long run Phillips curve is not vertical but instead

$$1/LRS|_{\text{Nimark (2008)}} = \lim_{k\to\infty}\frac{\theta}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\left(1-\beta\frac{\pi_{k+1}}{\pi_k}\right)+\frac{1}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\frac{\zeta_k}{\pi_k} \tag{63}$$

$$= \frac{\theta}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\left(1-\beta\lim_{k\to\infty}\frac{\pi_{k+1}}{\pi_k}\right) \tag{64}$$

which is identical to the slope under sticky prices from (23), recalling the definition of κ.

To see the decisiveness of this forecast/prediction error factorization, consider the sticky information Phillips curve (28) again, but we will stay in the time domain now

$$\pi_t - \frac{1-\lambda}{\lambda}\xi y_t = (1-\lambda)\sum_{i=0}^{\infty}\lambda^i E_{t-i-1}[\pi_t + \xi(y_t - y_{t-1})] \tag{65}$$

$$= (1-\lambda)\sum_{i=0}^{\infty}\lambda^i (E_{t-i-1}[\pi_t] - \pi_t + \xi(E_{t-i-1}[y_t] - y_t - E_{t-i-1}[y_{t-1}] + y_{t-1})) \tag{66}$$

$$+ (1-\lambda)\sum_{i=0}^{\infty}\lambda^i(\pi_t + \xi y_t - \xi y_{t-1}) \tag{67}$$

$$= \pi_t + \xi y_t - \xi y_{t-1} - \sum_{i=0}^{\infty}\lambda^i (E_{t-i}[\pi_t] - E_{t-i-1}[\pi_t]) \tag{68}$$

$$-\xi\sum_{i=0}^{\infty}\lambda^i(E_{t-i}[y_t] - E_{t-i-1}[y_t]) + \xi\sum_{i=1}^{\infty}\lambda^i(E_{t-i}[y_{t-1}] - E_{t-i-1}[y_{t-1}]) \tag{69}$$

which can be rearranged to yield

$$y_t = \lambda y_{t-1} + \frac{\lambda}{\xi}\sum_{i=0}^{\infty}\lambda^i(E_{t-i}[\pi_t] - E_{t-i-1}[\pi_t]) \tag{70}$$

$$+\lambda\sum_{i=0}^{\infty}\lambda^i(E_{t-i}[y_t] - E_{t-i-1}[y_t]) - \lambda\sum_{i=1}^{\infty}\lambda^i(E_{t-i}[y_{t-1}] - E_{t-i-1}[y_{t-1}]) \tag{71}$$

$$= \lambda y_{t-1} + v_t \tag{72}$$

where $v_t = \frac{\lambda}{\xi}\sum_{i=0}^{\infty}\lambda^i(E_{t-i}[\pi_t] - E_{t-i-1}[\pi_t]) + \lambda\sum_{i=0}^{\infty}\lambda^i(E_{t-i}[y_t] - E_{t-i-1}[y_t]) - \lambda\sum_{i=1}^{\infty}\lambda^i(E_{t-i}[y_{t-1}] - E_{t-i-1}[y_{t-1}])$ is a (or rather three) sequence(s) of forecast errors. Compare this to the frequency domain version in (39), expressed in the time domain using the inverse z-transform as stationary causal representations of y_t and π_t with respect to some shock ϵ_t ($y_t = \sum_{k=0}^{\infty} y_k\epsilon_{t-k} + \text{t.i.o.}\epsilon$ and $v_t = \sum_{k=0}^{\infty} v_k\epsilon_{t-k} + \text{t.i.o.}\epsilon$)

$$y_k = \lambda y_{k-1} + \underbrace{\frac{1}{2\pi i}\oint_{|z|=1} z^{k-1}\left(\frac{\lambda}{\xi}\pi(\lambda z) + \lambda y(\lambda z) - \lambda y(\lambda z)\lambda z\right)dz}_{v_k} \tag{73}$$

That is, the long run relation given by the sticky information model is $y_t = \lambda y_{t-1} + v_t$ with infinite sequences of forecast errors producing a time varying relation that disappears in the limit. Analogously, the long run relation given by Nimark's (2008) Phillips curve is given by the long run dynamic relation of the standard sticky price model to which he added the information imperfection. Setting the Calvo parameter θ to zero in (54) gives the Phillips curve with only the information imperfection

$$(1 - H_s)\pi_t = mc_t \Rightarrow y_t = \frac{1}{\Theta}\tilde{\zeta}_t \tag{74}$$

where $\tilde{\zeta}_t = \lim_{\theta \to 0} \zeta_t = \int (\pi_t - E[\pi_t | \mathscr{I}_s(j)]) dj$. That is, Nimark's (2008) Phillips curve is non vertical solely due to and identically to the standard sticky price model. The information rigidity in isolation gives a vertical long run Phillips curve as the forecast/prediction error vanishes in the forecasting limit from the assumption of common knowledge of rationality. As Coibion and Gorodnichenko (2015b) demonstrate, the sticky information and noisy information models both relate their respective information rigidities to the same relationship between average forecast errors and prediction revisions, consistent with our assessment that both models share a common, vertical long run Phillips curve.

Proposition 5 (Long Run Phillips Curve Slope under Only Imperfect Common Knowledge)**.** *The (inverse) slope of the Phillips curve* $\frac{\partial y_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}$ *for the imperfect common knowledge Phillips curve without the addition of sticky prices (74) is*

$$\frac{1}{\Theta}\frac{\partial \tilde{\zeta}_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} \tag{75}$$

which, in the long run as $k \to \infty$, is

$$1/LRS|_{Only\ Imperfect\ Common\ Knowledge} = \lim_{k\to\infty}\frac{1}{\Theta}\frac{\partial \tilde{\zeta}_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} = 0 \tag{76}$$

3.3. Phillips Curves under Finite Informational Inertia

Both of the above models have the property that the Phillips curve becomes vertical in the forecasting limit and the output gap is necessarily closed, i.e. $\lim_{j\to\infty} E_t\left[y_{t+j}\right] = 0$. TO demonstrate that the verticality is behind the determinacy properties we will subsequently derive, we now introduce a generic Phillips curve that is specified only be its satisfying the following version of the natural rate hypothesis following Carlstrom and Fuerst (2002)[22]

$$E_{t-q}[y_t] = 0\ \forall t \tag{77}$$

in such a model, the output gap is necessarily closed on average due to the law of iterated expectations - the Lucas (1973) expectational Phillips curve (10) analyzed above satisfies

[22] Explicit examples are models that implement contracts of predetermined prices in the vein of Fischer (1977) with finite duration, including Andrés, López-Salido, and Nelson's (2005, p. 1034) "Sticky information, staggered á la Taylor," as found also in Koenig (2004), Collard, Dellas, and Smets (2009), and Woodford (2010); the Mussa-McCallum-Barro-Grossmann "P-bar model"—see McCallum (1994) and McCallum and Nelson (2001); models of finitely staggered predetermined prices such as Fischer (1977) and Blanchard and Fischer (1989, pp. 390–394); Carlstrom and Fuerst's (2002, p 81-82) model in this spirit; as well as the expectational Phillips curve of Lucas (1973)—see also Sargent and Wallace (1975)—that formalized the rational expectations revolution here in (10).

this at $q = 1$. With the long run setting it at some finite horizon,[23] this ensures that Lucas's (1972) NRH is fulfilled. Such a supply side can be expressed as

$$y_t = \sum_{j=0}^{q-1} \left(E_{t-j}[y_t] - E_{t-j-1}[y_t] \right) = \sum_{j=0}^{q-1} \eta_{t-j}^{(t)} \tag{78}$$

Non-zero output gaps can be represented wholly as innovations or forecast errors, $\eta_{t-j}^{(t)}$ without making any conjecture as to admissible solutions, in the words of Friedman (1977, p. 456), "[o]nly surprises matter." Note that the effect of a surprise need not disappear immediately after impacting the output gap, it can have a lasting—but not permanent—effect. That is, there can be a short-run tradeoff between the output gap and inflation, but this tradeoff must not be permanent if the model is to satisfy the NRH.

Proposition 6 (Long Run Phillips Curve Slope under Finite Informational Inertia)**.** *The (inverse) slope of the Phillips curve $\frac{\partial y_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}$ for the finite informational inertia Phillips curve (78) is*

$$\begin{cases} \frac{\partial \eta_t^{(t+k)}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1} & \text{if } k < q-1 \\ 0 & \text{otherwise} \end{cases} \tag{79}$$

which in the long run, as $k \to \infty$ (and hence $k \geq q$) is

$$1/LRS|_{Informational\ Inertia} = 0 \tag{80}$$

3.4. Long Run Phillips Curves and the Natural Rate Hypothesis

The unspecified supply side (78) is derived from the NRH and here we will provide more insight into this hypothesis and its relevance to the specification of the supply sides above. In particular, how the sticky-price Phillips curve (8) violates the NRH by positing same dynamic tradeoff at every expectational horizon. In contrast, all three models of informational inertia examined here have in common that their Phillips curves become vertical in the long run, see table 1. This is decisive for the determinacy properties and we show how some standard interpretations (e.g., steady state) or alterations (e.g., indexation) to reconcile the sticky-price Phillips curve with the NRH are insufficient to remove a trade off that distorts the determinacy limits on monetary policy - the Taylor principle that we will show is identical across models under the NRH.

[23]It makes no difference for the conclusions that follow whether the long run sets in after four quarters or four millennia: q is completely arbitrary for the analysis so long as it is finite.

Phillips Curve	**Short Run Slope** (k) $\frac{\partial y_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}$	**Long Run Slope** ($k \to \infty$) $\lim_{k\to\infty}\frac{\partial y_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}$
	Sticky Price	
$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t$	$\frac{1}{\kappa}\left(1-\beta\frac{\partial \pi_{t+k+1}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}\right)$	$\frac{1}{\kappa}\left(1-\beta \lim_{k\to\infty}\frac{\partial \pi_{t+k+1}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}\right)$
	Frictionless Model	
$y_t = 0$	0	0
	Lucas's (1973) *Expectational Phillips Curve*	
$y_t = \alpha(\pi_t - E_{t-1}\pi_t)$	α if $k=0$ 0 otherwise	0
	Mankiw and Reis's (2002) Sticky Information	
$\pi_t = \frac{1-\lambda}{\lambda}\xi y_t + (1-\lambda)\sum_{i=0}^{\infty}\lambda^i E_{t-i-1}[\pi_t + \xi(y_t - y_{t-1})]$	$\left(\frac{\lambda^{k+1}}{1-\lambda^{k+1}}\frac{1}{\xi}\left(\sum_{j=0}^{k}\frac{\partial \pi_{t+j}}{\partial \epsilon_t}\right)\right)\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}$	0
	Informational Inertia	
$y_t = \sum_{j=0}^{i-1}\left(E_{t-j}[y_t] - E_{t-j-1}[y_t]\right)$	$\frac{\partial(E_t[y_{t+k}] - E_{t-1}[y_{t+k}])}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}$ if $k<i$ 0 otherwise	0
	Nimark's (2008) Imperfect Common Knowledge	
$\pi_t = (1-\theta)(1-\beta\theta)\Theta\sum_{k=0}^{\infty}(1-\theta)^k y_{t\|t}^{(k)} + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k \pi_{t+1\|t}^{(k+1)}$	$\frac{1}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\left[\theta\left(\frac{\partial \pi_{t+k}}{\partial \epsilon_t} - \beta\frac{\partial \pi_{t+k+1}}{\partial \epsilon_t}\right) + \frac{\partial \zeta_{t+k}}{\partial \epsilon_t}\right]\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}$	$\frac{\theta}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\left[1-\beta\lim_{k\to\infty}\frac{\partial \pi_{t+k+1}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}\right]$
	Nimark's (2008) Imperfect Common Knowledge - Only Information Rigidity	
$\pi_t = \Theta\sum_{k=0}^{\infty} y_{t\|t}^{(k)}$	$\frac{1}{\Theta}\frac{\partial \zeta_{t+k}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}$	0
	Nimark's (2008) Imperfect Common Knowledge - Only Price Rigidity	
$\pi_t = \frac{(1-\theta)(1-\beta\theta)}{\theta}\Theta y_t + \beta E_t \pi_{t+1}$	$\frac{\theta}{(1-\theta)(1-\beta\theta)}\left(1-\beta\frac{\partial \pi_{t+k+1}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}\right)$	$\frac{\theta}{(1-\theta)(1-\beta\theta)}\frac{1}{\Theta}\left[1-\beta\lim_{k\to\infty}\frac{\partial \pi_{t+k+1}}{\partial \epsilon_t}\left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t}\right]^{-1}\right]$

TABLE 1. (Inverse) Phillips Curve Slope

The NRH, succinctly by Friedman (1968, p. 11) "there is always a temporary trade-off between inflation and employment; there is no permanent trade-off[,]" postulates that the output gap is closed on average regardless of monetary policy. The hypothesis and associated vertical Phillips curve are central to the rational expectations revolution.[24] The NRH enjoys near-universal agreement,[25] in part as McCallum (1998, p. 359) points out that models that violate the NRH possess the "a priori implausible" implication that there exist inflationary paths on which "a nation can enrich itself in real terms permanently."

McCallum (2004, pp. 21–22) explicitly highlights that the standard New Keynesian Phillips curve (8) violates the NRH and draws a distinction between "Friedman's weaker version" and the "stronger Lucas version" of the NRH. The former states that a higher, but constant, rate of inflation cannot permanently affect output and the latter that no path for prices, inflation, inflation growth, etc., can permanently keep output above its natural level. While the distinction is doubtless appropriate from the perspective of the accelerationist controversy, the nomenclature of McCallum (2004) is perhaps misleading as Friedman (1977, p. 274) himself made explicit that his view of the NRH is not limited to such an accelerationist view: "[S]ome substitute a stable relation between the acceleration of inflation and unemployment for a stable relationship between inflation and unemployment—aware of but not concerned about the possibility that the same logic that drove them to a second derivative will drive them to even higher derivatives." In any event, Lucas's (1972) is the version that McCallum (1994) argues should be upheld by monetary models—the critique repeated more directly in McCallum (1998, p. 359).

First, one can confirm that the standard New Keynesian sticky-price model with Calvo (1983)-style overlapping contracts (8) cannot satisfy Lucas's (1972) NRH by simply taking unconditional expectations

$$E\left[y_t\right] = \frac{1}{\kappa}\left(E\left[\pi_t\right] - \beta E\left[\pi_{t+1}\right]\right) \neq 0 \tag{81}$$

[24]See Lucas (1972) and Sargent (1973), with Sargent (1987b, p. 7) calling Friedman's (1968) address the revolution's "opening shot" and Modigliani (1977, p. 5) deeming Lucas's (1972) rational-expectations version the "death blow to the already badly battered Keynesian position."

[25]By the late '70s, Friedman (1977, p. 459) could note that his and Phelps's (1967) NRH was so widely accepted that, as McCallum (2004, p. 21) remarks, "by 1980 even self-styled Keynesian economists were agreeing." Krugman (1994, p. 52) confirms that "[t]he natural rate hypothesis has received near-universal acceptance" and underlines that it "has a very solid basis in experience," an agreement that Phelps (1994, p. 81) was "delighted to see." Bernanke (2003) went a step further: "Friedman's [point ...] that long run output is determined entirely by real factors [...] is universally accepted today by monetary economists."

Even in the extreme parameterization of $\beta = 1$, the unconditional expectation of the output gap would still be nonzero ($E[y_t] \neq 0$) with nonstationary inflation. As made explicit by McCallum (1998, p. 359), the NRH requires that "$E[y_t] = 0$ for any monetary policy[;]" i.e., the unconditional expectation of the output gap must be zero for any monetary policy. Nothing in this statement limits this to steady state or stationary policies and, indeed, McCallum and Nelson (2010, p. 7) note that the "Lucas version [...] pertains to inflation paths more general than steady states." Note, furthermore, that (8) posits the same immutable tradeoff at every expectational horizon:

$$E_t\left[y_{t+j}\right] = \frac{1}{\kappa}\left(E_t\left[\pi_{t+j}\right] - \beta E_t\left[\pi_{t+j+1}\right]\right), \; \forall j \geq 0 \tag{82}$$

the tradeoff in the sticky-price model is so stable that, from the perspective of today, the same dynamic tradeoff is expected to exist unchanged into the infinite future. The only way for this Phillips curve to satisfy the NRH, is if $\kappa \to \infty$, making the Phillips curve always, so at every expectational horizon, vertical.

The sticky-price Phillips curve indexed either to steady-state inflation, see Yun (1996),

$$y_t = \frac{1}{\kappa}\left(\pi_t - \bar{\pi} - \beta(E_t[\pi_{t+1}] - \bar{\pi})\right) \tag{83}$$

or past inflation, see Christiano, Eichenbaum, and Evans (2005) for $\gamma = 1$ and Smets and Wouters (2003) for $0 < \gamma \leq 1$,

$$y_t = \frac{1}{\kappa}\left(\pi_t - \beta E_t[\pi_{t+1}] - \gamma\left(\pi_{t-1} - \beta\pi_t\right)\right) \tag{84}$$

still fails to satisfy Lucas's (1972) NRH,[26] for the same reason above. Only those monetary policies that lead to a stationary path for inflation allow the output gap to be equal, on average, to zero. As above, these Phillips curves can be made to satisfy the NRH, but this requires $\kappa \to \infty$, making them always vertical.

Requiring the equilibrium path of a linearized system to be bounded does not mean that nonstationary paths are inconsequential for local analyses. The local determinacy of a bounded equilibrium path in the linearized system depends crucially on all other potential paths becoming unbounded so that this bounded path is unique. These other paths need never materialize: their mere hypothetical existence in the stead of additional bounded paths that could not be excluded is what renders the single bounded equilibrium unique. This is Cochrane's (2011) assessment of determinacy through a Taylor rule being an off-equilibrium threat and requiring determinacy imposes bounds on coefficients

[26] See also McCallum (2004, pp. 21–22) and McCallum and Nelson (2010, pp. 6–7).

in a policy rule to ensure that there is a unique locally bounded equilibrium. That is, determinacy rests on the ability to convincingly predict when an equilibrium path would become unbounded such that it can be excluded from the class of permissible equilibria; that arguments resting on permanent output-inflation tradeoffs are not convincing is, of course, a central component of the NRH. Reinterpreting the New Keynesian Phillips curves in terms of output gaps driven by inflation makes the violation of the NRH and its consequences for determinacy more visible: the right-hand sides of (8), (83), and (84) present a description of the dynamic properties of inflation to achieve any desired dynamic for the output gap. Of course for determinacy analysis, it is unbounded dynamics that are desired for all but one equilibrium paths to render this remaining equilibrium uniquely bounded. Thus, it is not the final equilibrium under study that would display the aberrant behavior implied by an exploitation of these Phillips curves, but rather the hypothetical paths that are being excluded for the sake of equilibrium uniqueness; and these Phillips curves show indeed that inflation and the output gap can work in concert to such an end all the way through to the long run. A NRH model, in contrast, must display a vertical Phillips curve that prevents any such concerted long-run reaction: only unexpected components of inflation cause output gaps.

4. Existence and Uniqueness for Sticky Information

Now we turn to establishing the determinacy bounds with the Taylor rule (1), the dynamic IS equation (2), and the three different supply curves characterized by informational imperfections. Note that the absence of exogenous driving forces is without loss of generality and will remain the same if our systems are appended with stationary driving forces (i.e., we are investigating the properties of the homogenous component of the system of difference equations).[27] For a complete solution, one would then have the additional task of associating the exogenous driving forces with the expectation errors (see, e.g., Sims (2001)). This is precisely the advantage of our analysis, we separate the question of whether there is a unique equilibrium from what this equilibrium is.

Beginning with the sticky information model and, see the previous section,

$$\frac{\xi}{\lambda}y(z) = z\xi y(z) + \pi(\lambda z) + \xi(1-\lambda z)y(\lambda z) \tag{85}$$

its Phillips curve is recursive in the frequency domain. We will exploit this and establish conditions for its determinacy in the frequency domain. To make the connection to

[27] See footnote 9 and also footnote 11.

standard, time domain results more clear, we reestablish the determinacy conditions but in the frequency domain for the sticky price model in the appendix. With the supply side in the frequency domain, we close the model by combining the Taylor rule (1), the dynamic IS equation (2) and expressing in the frequency domain to give

$$(1+\sigma\phi_y)zy(z)+\sigma\phi_\pi z\pi(z)=y(z)-y_0+\sigma(\pi(z)-\pi_0) \tag{86}$$

Notice that we are abstracting from shocks and these equations (along with a supply curves from above) are entirely homogenous. Thus one solution, the fundamental solution is zero at all frequencies - an inability to rule out nonzero solutions is tantamount to not being able to rule out stable sunspot solutions - i.e. non-uniqueness or indeterminacy. We summarize determinacy in the following.

Theorem 4 (Sticky Information Determinacy)**.** *The sticky information model, given by (2), (85), with the Taylor rule (1), has a unique, stable equilibrium if and only if*

$$\phi_\pi > 1 \tag{87}$$

So, $\phi_\pi > 1$ is a necessary condition for determinacy in the sticky information model and not merely sufficient as above in the sticky price model. No amount of output gap targeting can replace a more than one for one response to inflation by the monetary authority. As with the frictionless and Lucas's (1973) expectational Phillips curves, the Taylor principle as a policy recommendation holds directly.

Under a simple, current inflation-targeting rule, determinacy is obtained if the central bank follows an active monetary policy satisfying the Taylor principle. This is true for both the sticky price and the sticky information model. Including output gap targeting into the leads to different consequences for determinacy in the two models. In the presence of sticky prices, the a reaction to inflation and/or the output gap can achieve stability. Output gap movements are translated into inflation movements at a rate of $(1-\beta)/\kappa$ - the Phillips curve relationship in the long run. With a vertical long run Phillips curve under sticky information model, such a translation is not possible and monetary policy must respond to inflation directly to satisfy the Taylor principle.

5. Existence and Uniqueness for Imperfect Common Knowledge

The model of imperfect common knowledge by Nimark (2008) combines an information rigidity with the sticky price rigidity. This is particularly insightful as with both rigidities, the determinacy bounds will coincide with the sticky price results from theorem 2 and with only the information rigidity with those of the frictionless model in theorem 1.

Recall the imperfect common knowledge Phillips curve (55)

$$\pi_t = \beta E_t \pi_{t+1} + \kappa y_t + \zeta_t \tag{88}$$

where the imperfect common knowledge and higher order expectations having been encapsulated in forecast/prediction errors ζ_t. Observe that the model is identical to that of theorem 2 apart from the forecast/prediction error ζ_t that, like exogenous driving forces we have abstracted from, are irrelevant for determinacy.[28]

Theorem 5 (Nimark (2008) Determinacy)**.** *The imperfect common knowledge sticky price model, given by (2), (55), and (1), has a unique, stable equilibrium if and only if*

$$1 - \frac{1-\beta}{\kappa}\phi_y < \phi_\pi \tag{90}$$

The imperfect common knowledge Phillips curve has as a special form (74) with only the information rigidity, i.e. without the sticky price friction of (74)

$$y_t = \tilde{\zeta}_t \tag{91}$$

which is analogous to (3) in the frictionless model but now with the prediction/forecast innovation $\tilde{\zeta}_t$, and hence we obtain the same determinacy restriction

Theorem 6 (Imperfect Common Knowledge Determinacy)**.** *The imperfect common knowledge model, (2), (74), and (1), has a unique, stable equilibrium if and only if*

$$1 < \phi_\pi \tag{92}$$

Comparing theorem 6 to theorem 4 allows us to conclude that the determinacy results under sticky information and imperfect common knowledge are identical and comparing these further to theorem 5 and theorem 2 we remark that the instilling a model of imperfect information with a long run tradeoff from, say, a Calvo sticky price friction will instill it with the same determinacy properties of the latter.

[28] We reiterate that providing the condition under which we have a determinant solution is not the same as providing the solution. As before, see footnote 32, calculating the solution in the presence of exogenous shocks would require us to resolve the prediction/forecast errors in ζ_t. Solving forward (H.50) yields

$$\begin{bmatrix} \pi_t \\ y_t \end{bmatrix} = \lim_{j\to\infty} A^{-j} \begin{bmatrix} E_t \pi_{t+j} \\ E_t y_{t+j} \end{bmatrix} + \sum_{j=0}^{\infty} A^{-j} \begin{bmatrix} \frac{1}{\beta} \\ 0 \end{bmatrix} E_t \zeta_{t+j} \tag{89}$$

and this sequence of prediction/forecast errors $E_t\zeta_{t+j}$ would have to be resolved consistent with the exogenous shocks and conditioning assumptions on agents' information sets.

6. Existence and Uniqueness for Finite Inattentiveness

To address determinacy of equilibria under the finite informational inertia supply side in (78), we will first assess determinacy in the more general class that includes our model

$$0 = \sum_{i=0}^{p} \sum_{j=-m}^{n} Q(i,j) E_{t-i} X_{t+j}, \;\; X_t = \begin{bmatrix} R_t & \pi_t & y_t \end{bmatrix}', \;\; 0 \le p, m, n < \infty \tag{93}$$

where the $Q(i,j)$'s are matrices of dimensions 3×3. I.e., the model is composed of three structural equations: the supply side, the demand side, and monetary policy. The class encompasses all linear rational expectations models in the three variables of interest that (i) have a finite number of leads (given by n), (ii) have a finite number of lags (given by m), and (iii) have expectations formed at horizons from t into the finite past $t-p$.[29] This captures a wide range of interest rate rules found in the literature, from our standard Taylor rule (1) to the variety of extensions we will examine later.

Theorem 7. *For the system (93) to have a unique stationary solution,*

(1) The perfect foresight version of (93)

$$0 = \sum_{j=-m}^{n} \tilde{Q}_j X_{t+j} \tag{94}$$

where $\tilde{Q}_j = \sum_{i=0}^{p} Q(i,j)$, must have a unique saddle-point stable solution.

(2) The square matrix

$$\begin{bmatrix} \mathbf{Q}' & \mathbf{B}' \end{bmatrix}' \tag{95}$$

must be non-singular. $\mathbf{Q}$ and $\mathbf{B}$ are block matrices of dimensions $3p \times 3(p+n)$ and $3n \times 3(p+n)$ respectively with blocks of dimension 3×3. The s^{th} block row of $\mathbf{Q}$ is

$$\begin{bmatrix} 0_{max(0,s-1-m)} & \tilde{Q}(s-1, -\min(s-1,m), n) & 0_{p-s} \end{bmatrix} \tag{96}$$

where 0_i is a $3 \times 3i$ matrix of zeros and $\tilde{Q}(a,b,c) = \begin{bmatrix} \tilde{Q}(a,b) & \tilde{Q}(a,b+1) & \ldots \tilde{Q}(a,c) \end{bmatrix}$ with $\tilde{Q}(a,b) = \sum_{i=0}^{\min p,a} Q(i,b)$. The s^{th} block row of $\mathbf{B}$ is

$$\begin{bmatrix} 0_{max(0,s+p-m-1)} & -\tilde{B}(\min(p+s-1,m)) & I & 0_{n-s} \end{bmatrix} \tag{97}$$

[29] As discussed previously, the absence of exogenous driving forces in (93) is without loss of generality. The conditions for determinacy remain the same if (93) is appended with stationary driving forces (i.e., we am investigating the properties of the homogenous component of the system of difference equations). For a complete solution, one would then have the additional task of associating the exogenous driving forces with the expectation errors (see, e.g., Sims (2001)).

where I is a 3×3 identity matrix and $\tilde{B}(a)$ the last $3 \times 3a$ elements of the $3 \times 3m$ matrix B of Anderson's (2010, p. 7) convergent autoregressive solution to (94).

The first condition requires that the model is determinate if the information rigidity were removed and the second requires that one can uniquely resolve the prediction errors. The first is Anderson's (2010) extension of the familiar Blanchard and Kahn (1980) result, while the second formalizes Whiteman's (1983, pp. 29–36) insight that resolving lagged expectations, "withholding constraints" in his terminology, is not generally a trivial task. This second restriction will hold generically unless the model of informational inertia is ill-specified such that at some intermediate forecasting horizon there is a(n) (un)fortuitous collinearity with another equation or forecasting horizon., i.e., due to a singularity of the matrix $\begin{bmatrix}\mathbf{Q}' & \mathbf{B}'\end{bmatrix}'$ - see footnote 10 and the simple example in the online appendix. Hence, although it is a non trivial task to resolve these constraints numerically, as the literature on such models has clearly demonstrated, it is the first restriction that is relevant for determinacy and monetary policy. Theorem 7 establishes the determinacy conditions under the finite informational inertia supply curve (78) as summarized in the following

Theorem 8 (Determinacy in the Model of Finite Inattentiveness)**.** *The finite informational inertia model, given by (2), (78), and (1), has a unique, stable equilibrium only if*

$$1 < \phi_\pi \tag{98}$$

Proof. The result is an immediate consequence of the first condition of theorem 7, recognizing that the frictionless equivalent of (78) is $y_t = 0$, and appealing to theorem 1. □

Recall from above that the "if" is only missing due to the possibility of a(n) (un)fortuitous collinearity in the information rigidity. Excepting this, we conclude that the determinacy conditions for our model of finite informational inertia, imperfect common knowledge (*without* additional sticky price rigidities) and sticky information are all identical and coincide with the determinacy results in a frictionless model, i.e. theorem 1.

All three of the informational inertia models we examined - sticky information, imperfect common knowledge, and finite informational inertia - share a common vertical long run Phillips curve, as we contrast in table 1. This means that the more than one for one response of the nominal interest rate in response to inflation of the Taylor principle can only be satisfied in the long run directly - the degree of output gap targeting is irrelevant for determinacy as it is necessarily closed in the long run with a vertical Phillips curve. In the sticky-price New Keynesian model, the NRH does not hold at any horizon and its long run Phillips curve posits a stable dynamic tradeoff. As a consequence, the sticky-price model is not even asymptotically isomorphic to its frictionless equivalent: with this permanent link a response of the interest rate to the output gap is equivalent (proportional to the slope of the long run Phillips curve) to a response to inflation. That is, the Taylor principle does not need to be satisfied directly and output gap targeting can substitute for inflation targeting with respect to determinacy.

7. General Forms of the Taylor Rule

Here we extend our results to more general specifications of the Taylor rule. First a rule with arbitrary horizons of inflation, output gap, and output growth targeting and second with a rule that incorporates interest rate smoothing. The first strengthens our conclusion that determinacy bounds are more conservative with a vertical long-run Phillips curve with not only no degree of any output or output growth targeting at any horizon being relevant for determinacy but also no degree of inflation targeting at any horizon other than the present relevant for determinacy. Then turning to interest rate smoothing, we find that the inertia in the interest rate imbues it with sufficient history dependence to open a window for inflation expectation targeting to be associated with a unique stationary equilibrium - a window because although the Taylor principle continues to be necessary, now an upper bound for the degree of inflation targeting appears. Finally, we find Davig and Leeper's (2007) “long-run” Taylor principle holds in its straightforward form and no longer supports their result under sticky prices that calls the indeterminacy result of Lubik and Schorfheide (2004) for US monetary policy of the 1970s into question.

7.1. Arbitrary Leads and Lags in the Taylor Rule

Consider first the following rule with arbitrary targeting horizons

$$R_t = \phi_\pi E_t \pi_{t+j} + \phi_y \left(\alpha_y E_t y_{t+m} + \left(1-\alpha_y\right) E_t \Delta y_{t+m}\right) \tag{99}$$

j and m captures targeting inflation and real activity at different horizons and α_y the level ($\alpha_y = 1$) as well as growth ($\alpha_y = 0$) of real activity.

Theorem 9 (Inattentiveness and the General Taylor Rule)**.** *An informational inertia model, given by (28), (74), or (78) on the supply side; (2) on the demand side; and the general Taylor rule (99) for monetary policy has a unique, stable equilibrium if and only if*

$$\phi_\pi > 1 \textit{ and } j = 0 \tag{100}$$

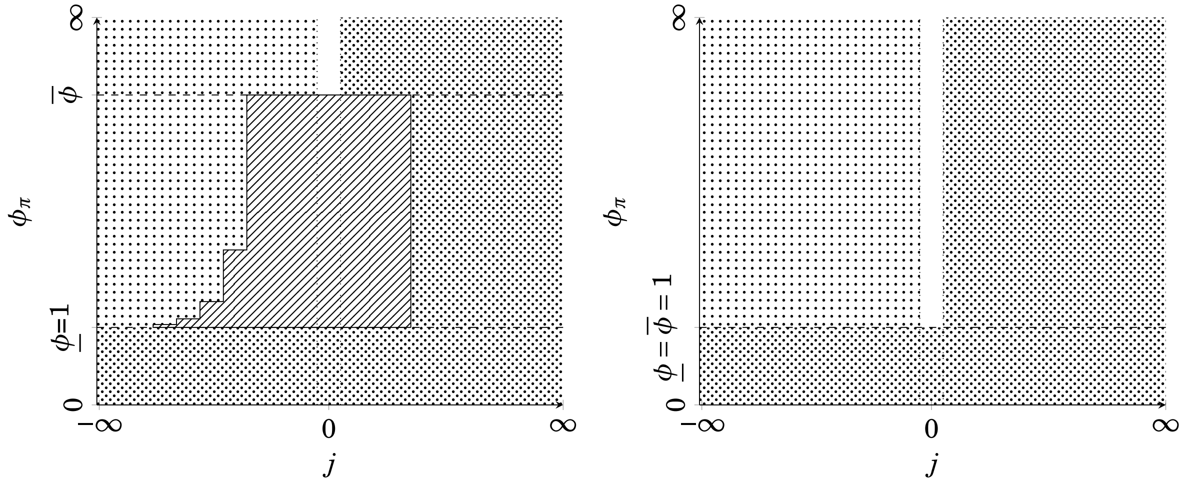
Note that theorem 9 contrasts starkly with existing results in sticky prices, see table 2. Examining the table, which contains several different variants of Taylor rules examined for determinacy in the literature as special cases of theorem 9 for models of informational inertia, the first thing to notice is the utter simplicity of the results under information rigidities. No model specific coefficients, such as subjective discount factors, degree of nominal or informational rigidities, no elasticity of intertemporal substitution is needed to ascertain the restrictions on the monetary policy rule. This is particularly appealing in an uncertain environment, where these parameters are likely to be known only with limited precision. Note, following section 3.4, that setting $\beta = 1$, does not render the bounds identical in the sticky price and information models: a long-run *dynamic* tradeoff remains $\frac{\pi_t - E_t \pi_{t+1}}{\kappa} = y_t$ which opens the possibility of monetary policy targeting past or future inflation (i.e., backward or forward looking targeting) - but even then these are not complete substitutes as they face upper bounds for the reaction to (past/future) inflation. Lubik and Marzo (2007) reconcile this result with non monotonic (e.g., oscillating) sunspot dynamics in the sticky price model - the sticky information model admits no such possibility, just as neither output gap nor growth targeting cant replace a concern for inflation, so too can neither a concern for past nor for future inflation replace the necessity of the monetary authority to vigorously respond to current inflation.

Taking a closer look, the restrictions implied by informational inertia are again more conservative than under sticky prices: if determinacy is given under informational inertia, it also implies determinacy under sticky prices. Hence, in the face of model uncertainty, a policy maker with a concern for robustness would be well-advised to heed the restrictions we provide here. The restrictions are far from being obscure and in fact are straightforward: the celebrated Taylor principle is necessary and sufficient for determinacy. Yet, it is the Taylor principle in its perhaps simplest, but certainly most direct form that is relevant: the policy rule must posit a contemporaneous, more than one-for-one direct response of the nominal interest rate to inflation. An indirect response via the output gap or its growth rate is insufficient - concern for the real economy can not replace a concern for inflation. This is only possible in the sticky price model as it posits a stable long run tradeoff between inflation and the output gap. This tradeoff is absent in models of informational inertia as we have reiterated in the analysis above and hence the measure of the monetary authority's rule is in its direct response to current inflation.

Taylor Rule	Informational Inertia	Sticky Price[a]	
		Lower Bound	Upper Bound
Contemporaneous			
$R_t = \phi_\pi \pi_t$	$1 < \phi_\pi$	$1 < \phi_\pi$	$\varnothing$
$R_t = \phi_\pi \pi_t + \phi_y y_t$	$1 < \phi_\pi$	$\max\left\{1 - \frac{1-\beta}{\kappa}\phi_y, 0\right\} < \phi_\pi$	$\varnothing$
Forward looking			
$R_t = \phi_\pi E_t \pi_{t+1}$	$\phi_\pi = \varnothing$	$1 < \phi_\pi$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma}$
$R_t = \phi_\pi E_t \pi_{t+1} + \phi_y y_t$	$\phi_\pi = \varnothing$	$\max\left\{1 - \frac{1-\beta}{\kappa}\phi_y, 0\right\} < \phi_\pi$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma} + \frac{1+\beta}{\kappa}\phi_y$
$R_t = \phi_\pi E_t \pi_{t+1} + \phi_y E_t y_{t+1}$	$\phi_\pi = \varnothing$	$\max\left\{1 - \frac{1-\beta}{\kappa}\phi_y, 0\right\} < \phi_\pi$, $0 \le \phi_y < 1/\sigma$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma} - \frac{1+\beta}{\kappa}\phi_y$, $0 \le \phi_y < 1/\sigma$
$R_t = \phi_\pi E_t \pi_{t+1} + \phi_y E_t \Delta y_{t+1}$	$\phi_\pi = \varnothing$	$1 + \phi_y(1+\beta+\kappa) + \frac{1+\kappa+\beta}{\sigma} < \phi_\pi$, $1/\sigma < \phi_y$	$\phi_\pi < 1 + \frac{\kappa+\beta}{\sigma} - \phi_y(1+\kappa+\beta)$, $1/\sigma < \phi_y$
Backward-looking			
$R_t = \phi_\pi \pi_{t-1}$	$\phi_\pi = \varnothing$	$1 < \phi_\pi$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma}$
$R_t = \phi_\pi \pi_{t-1} + \phi_y y_{t-1}$	$\phi_\pi = \varnothing$	$\max\left\{1 - \frac{1-\beta}{\kappa}\phi_y, 0\right\} < \phi_\pi$, for $0 \le \phi_y < \frac{1+\beta}{\sigma\beta}$ or $\max\left\{1 + 2\frac{1+\beta}{\kappa\sigma} - \frac{1+\beta}{\kappa}\phi_y, 0\right\} < \phi_\pi$, for $\frac{1+\beta}{\sigma\beta} < \phi_y$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma} - \frac{1+\beta}{\kappa}\phi_y$, for $0 \le \phi_y < \frac{1+\beta}{\sigma\beta}$ or $\phi_\pi < \max\left\{1 - \frac{1-\beta}{\kappa}\phi_y, 0\right\}$, for $\frac{1+\beta}{\sigma\beta} < \phi_y$
Interest rate smoothing			
$R_t = \rho_R R_{t-1} + (1-\rho_R)[\phi_\pi E_t \pi_{t+1} + \phi_y y_t]$	$1 < \phi_\pi < \frac{1+\rho_R}{1-\rho_R}$	$\max\left\{1 - \rho_R - \frac{1-\beta}{\kappa}(1-\rho_R)\phi_y, 0\right\} < \phi_\pi$, $0 \le \rho_R < \beta$	$\phi_\pi < 1 + 2\frac{1+\beta}{\kappa\sigma} + \frac{1+\beta}{\kappa}(1-\rho_R)\phi_y + \left(1 + 2\frac{1+\beta}{\kappa\sigma}\right)\rho_R$, $0 \le \rho_R < \beta$

TABLE 2. Determinacy Bounds on Monetary Policy

[a] See Bullard and Mitra (2002) and Lubik and Marzo (2007).



(A) Sticky Price, $\underline{\phi} = 1, \overline{\phi} = 1 + 2\frac{1+\beta}{\kappa\sigma}$ (Loisel, 2022) (B) Informational Inertia, $\underline{\phi} = \overline{\phi} = 1$

FIGURE 1. Determinacy Regions

□: Determinacy, ▨: Potential Determinacy, ⚄: Indeterminacy, ⚃: Nonexistence

Our theorem 9 is directly compatible with Loisel (2022), who provides and analysis of determinacy in a wide set of sticky price models from the Macroeconomic Model Data Base (MMB) (see Wieland, Cwik, Müller, Schmidt, and Wolters, 2012; Wieland, Afanasyeva, Kuete, and Yoo, 2016) [30] which includes backward looking New and "Old" Keynesian models with different dynamics in the long run tradeoffs and Taylor rules with arbitrary horizons of inflation targeting. The decisiveness of our restriction on monetary policy is again striking: with any horizon possible and inflation and/or output gap and growth targeting possible, determinacy is obtained if and only if the central bank responds to contemporaneous inflation more than one-for-one. Figure 1 depicts the situation, with our restriction in the lower panel and Loisel's (2022) for purely inflation targeting (again, a fleeting glance at table 2 ought to suffice to convince the reader that simultaneous inflation and output gap targeting at arbitrary horizons is likely to be a very complicated undertaking). It is the intermediate region between $\underline{\phi}$ and $\overline{\phi}$ in the upper panel of Loisel (2022) that constitutes the disagreement. Precisely the varying long run tradeoffs lead to the region of potential determinacy in the interior of the upper panel in his analysis. When these tradeoffs disappear in the long run, this interior region of potentially (dynamically) extended determinacy disappears: only a more than one for one response to current inflation provides determinacy as is depicted in the lower panel.

[30]See the Github and Website of the Rep-MMB that prepares the models of the MMB for comparisons.

The determinacy disagreement hinges on the slope of the long run Phillips cure. The sticky price model possess a vertical long run Phillips curve if and only if $\kappa \to \infty$ (also rendering its short run slope vertical). Our bounds under informational inertial models can be recovered from the sticky price restrictions simply by setting $\kappa \to \infty$ in our table 2. Hence, rejecting our conservative bounds on monetary policy to deliver a unique, stable equilibrium is not a consequence of preferring one New (or "Old") Kenyesian model over another, but rather of positing a stable long run tradeoff between output and inflation.

7.2. Interest Rate Smoothing

Consider now the rule with interest rate smoothing

$$R_t = \rho_R R_{t-1} + (1-\rho_R)\left[\phi_\pi E_t \pi_{t+j} + \phi_y \left(\alpha_y E_t y_{t+m} + \left(1-\alpha_y\right) E_t \Delta y_{t+m}\right)\right] \qquad (101)$$

$0 \leq \rho_R < 1$ allows for interest rate smoothing along with the generality of varying horizons and measures of real activity in (99).

Theorem 10 (Informational Inertia and the General Taylor Rule with Interest Rate Smoothing)**.** *An informational inertia model, given by (28), (74), or (78) on the supply side; (2) on the demand side; and the general Taylor rule with interest rate smoothing (101) for monetary policy has the following properties with respect to a stable equilibrium*

(1) *indeterminacy if* $\phi_\pi < 1$
(2) *indeterminacy if* $\frac{1+\rho_R}{1-\rho_R} < \phi_\pi$ *and* $j > 0$
(3) *nonexistence if* $\frac{1+\rho_R}{1-\rho_R} < \phi_\pi$ *and* $j < 0$
(4) *determinacy if* $1 < \phi_\pi$ *and* $j = 0$
(5) *determinacy if* $1 < \phi_\pi < \frac{1+\rho_R}{1-\rho_R}$ *and* $j = 1$

Again, we see more restrictive bounds on monetary policy than in the sticky price model (see table 2). There is, however, a broadening of the strict interpretation of the Taylor principle as the history dependence of monetary policy through interest rate smoothing implies responses to the contemporaneous inflation rate at differing horizons of inflation targeting. This can be seen via the simplified one period inflation horizon version $R_t = \rho_R R_{t-1} + (1-\rho_R)\left[\phi_\pi E_t \pi_{t+1}\right] = (1-\rho_R)\phi_\pi \left[E_t \pi_{t+1} + \rho_R E_{t-1} \pi_t + \ldots\right]$ which clearly imparts the interest rate rule with a concern for current inflation (precisely past expectations thereof). This broadening, however, is limited sharply by the degree of history dependance by the upper bound. As in the analysis of the sticky price model by Lubik and Marzo (2007), sunspots need not be monotonic or constant, but may also be oscillating and a too strong a response to future expected inflation in the presence of interest rate smoothing is consistent with such non monotonic sunspots. At higher horizons of future inflation expectations, this window of determinacy collapses and as with all rules, the degree of output gap targeting - level, growth, expectation, lagged - is irrelevant for determinacy.

7.3. Monetary Regimes

Accounting for evidence of regime shifts in monetary policy - see Sims and Zha (2006) and Hamilton (2016), Davig and Leeper (2007) extend the determinacy analysis to a Taylor with regime shifts in the reaction coefficients and replace (1) with

$$R_t = \phi_\pi(s_t)\pi_t + \phi_y(s_t)y_t \tag{102}$$

where $\phi_\pi(s_t)$ and $\phi_y(s_t)$ depend on the realized policy s_t, which is either 1 or 2, via

$$(\phi_\pi(s_t), \phi_y(s_t)) = \begin{cases} (\phi_{\pi,1}, \phi_{y,1}) & \text{for } s_t = 1 \\ (\phi_{\pi,2}, \phi_{y,2}) & \text{for } s_t = 2 \end{cases} \tag{103}$$

and the regime follows a Markov chain with probabilities $p_{ij} = P(s_t = j|s_{t-1} = i)$.

Davig and Leeper (2007) initially examine inflation determination by closing the model with the Fisher equation (4) or, equivalently, in the frictionless model with a dynamic IS equation (2) and a closed output gap (3), $y_t = 0$. From the preceding analyses of this paper, it is straightforward that their results from the frictionless setting carry over to our models of informational inertia with vertical long run Phillips curves.

Theorem 11 (Informational Inertia and Davig and Leeper's (2007) Generalized Taylor Principle)**.** *A model of informational inertia, given by (28), (74), or (78) on the supply side; (2) on the demand side; and the regime shifting Taylor rule (102) for monetary policy has a unique, stable equilibrium if and only if*

- $\phi_\pi(s_t) > 1$ *for some* s_t *and either*
  (1) the regime is fixed at this s_t *or*
  (2) $\big(1-\phi_\pi(2)\big)p_{11} + \big(1-\phi_\pi(1)\big)p_{22} + \phi_\pi(1)\phi_\pi(2) > 1$

abstracting from non-generic collinearity in the resolution of prediction errors - see, e.g., the discussion following Theorem 7.

So we see that the degree of output targeting, as in all of the above, as well as structural parameters outside the monetary policy rule (which here also include the transition probabilities of the monetary policy regimes) are irrelevant for determinacy also in the case of regime switching in monetary policy. Davig and Leeper (2007) denote the restriction in the theorem as the "long run" Taylor principle. There it is long run the sense that monetary policy might deviate from the fixed regime Taylor principle $\phi_\pi > 1$ occasionally (i.e., in some regimes) but still exclude nonfundamental sources of variation. Long run in the sense of this paper, and of course key to understanding the results we have presented, is in expectation from a particular point in time looking far enough (which may the limit in the case of sticky information of imperfect common knowledge for these informational rigidities to resolve) forward.

In their analysis within a sticky price New Keynesian model, Davig and Leeper (2007) find that recurring regime change makes determinacy depend on the policy process and all parameters, as they affect intertemporal margins that interact with expected policies. This hinges crucially on the sticky price model which posits such an intertemporal margin in the Phillips curve that is present at every horizon - if we accept the natural rate hypothesis, then we must reject the presence of such a margin in the long run (in our use of the term, in the current period's expectation of the limiting relationship) and satisfy the determinacy bounds from the frictionless model which are consistent with any of the models of the supply side above that have a vertical long run Phillips curve.

Davig and Leeper (2007) find in their sticky price New Keynesian analysis that inelastic intertemporal substitution by households and a nonzero probability of price nonadjustment expand the region of monetary policy elasticities with respect to inflation that are consistent with determinacy. Our results would caution policy makers from relying on this expanded region. Empirically, Davig and Leeper (2007) call the assessment of Lubik and Schorfheide (2004) into question that indeterminacy of monetary policy and the presence of nonfundamental variation were consistent with the US macroeconomic experience during the Great Inflation of the 1960s and 70s. With Lubik and Schorfheide's (2004) estimations of policy parameters pre- and post-Volker, Davig and Leeper (2007) calculate that the model was determinate despite the passive policy during the Great Inflation, taking the expected duration of regimes as five years. Our analysis is more straightforward - requiring only the regime transition probabilities and inflation elasticities of the monetary regimes - and questions their results: with $\phi_1 = 2.19$ in the post 1982 and $\phi_2 = 0.89$ during the 70s and $p_{11} = p_{22} = 0.95$, we have $\phi_1 > 1$

but $\left(1-\phi_\pi(2)\right)p_{11}+\left(1-\phi_\pi(1)\right)p_{22}+\phi_\pi(1)\phi_\pi(2)=0.9231<1$. That is, we find that indeterminacy may indeed still have been present despite the presence (and thus agents' expectations of) a stabilizing regime.

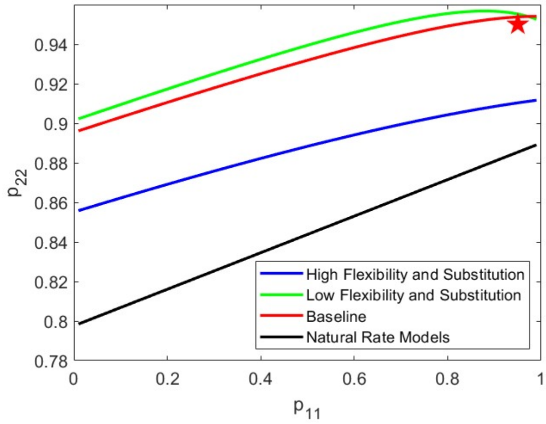


FIGURE 2. Determinacy Bound: Determinacy is obtained below the curves

We summarize and link the results to Davig and Leeper (2007) in figure 2. The curves give the maximal p_{22} associated with determinacy in four cases - that is the largest continuation probability of the "passive" monetary policy regime with $\phi_2 = 0.89$ that still rules out nonfundamental disturbances. The curve labeled "baseline" refers to Davig and Leeper's (2007) baseline New Keynesian model that led them to cast doubt on the indeterminacy hypothesis of pre Volker monetary policy and those labeled "high flexibility and substitution" and "low flexibility and substitution" are those used in Davig and Leeper's (2007) figure 4 and capture extreme values of the 90% posterior set from the estimates of Lubik and Schorfheide (2004). Hence, we see that Davig and Leeper's (2007) "baseline" is close to the extreme parameterization associated with the largest determinacy set. The curve labeled "natural rate models" is their bound for the frictionless model that our theorem 11 above extends to the various supply curves that have vertical long run Phillips curves here. Obviously, this bound is associated with less determinacy - i.e., in the absence of a long run tradeoff between output and inflation, Davig and Leeper's (2007) "intertemporal margins that interact with expected policies," monetary policy needs to be less likely to stay in the regime with passive policy than the sticky price model leads one to believe. Indeed, the red star in the figure gives the $p_{11} = p_{22} = 0.95$ from above with is clearly above the black line associated with our bound.

8. Fiscal and Monetary Interactions

Here we assess the consequences of our results for the intersection of monetary policy with fiscal policy. Turning Sargent and Wallace's (1981) monetary dominance on its head, Leeper (1991) shows that an active fiscal policy can generate a unique bounded equilibrium if monetary policy is passive - the price level adjusts endogenously to equate the real value of debt to the present value of future public surpluses. This is the fiscal theory of the price level - see Canzoneri, Cumby, and Diba (2010) and Cochrane (2023) for overviews and introductions - and an active or non-Ricardian fiscal policy, such as real primary surplus responding insufficiently to deviations in real debt for a given price level, generates fiscal dominance with inflation adjusting according to the mechanism described above. We follow McClung (2021) and adopt his simple model to get to the fiscal-monetary tradeoff as expediently as possible. Add to demand (2), monetary policy (1), and a supply curve the following intertemporal government budget constraint[31]

$$b_t = \beta^{-1}(b_{t-1} - \pi_t) + R_t - \tau_t \tag{104}$$

with b_t real government debt and τ_t real fiscal surpluses determined by the fiscal rule

$$\tau_t = \phi_d b_{t-1} \tag{105}$$

which has the fiscal authority adjusting surpluses in response to debt - a sufficiently vigorous response is characterized as "active" fiscal policy.

Leeper (1991) examines determinacy through monetary and fiscal policy with the Fisher equation (4) or, equivalently, in the frictionless model with a dynamic IS equation (2) and a closed output gap (3), $y_t = 0$. Combining with the interest rate rule for (1), the system of equations can be combined into one equation in the dynamics of debt

$$0 = E_t[b_{t+1}] - \left(\beta^{-1} - \phi_d + \phi_\pi\right) b_t + \phi_\pi \left(\beta^{-1} - \phi_d\right) b_{t-1} \tag{106}$$

Factor by defining λ_1 and λ_2 such that $0 = \lambda^2 - \left(\beta^{-1} - \phi_d + \phi_\pi\right)\lambda + \phi_\pi\left(\beta^{-1} - \phi_d\right) = (\lambda - \lambda_1)(\lambda - \lambda_2)$ and following Viete $\beta^{-1} - \phi_d + \phi_\pi = \lambda_1 + \lambda_2$ and $\phi_\pi\left(\beta^{-1} - \phi_d\right) = \lambda_1\lambda_2$, thus

$$0 = E_t[b_{t+1}] - (\lambda_1 + \lambda_2) b_t + \lambda_1\lambda_2 b_{t-1} \tag{107}$$

[31]Compare with Leeper's (1991) (3.2) which has the fiscal rule that follows already substituted in, Woodford's (2003) (4.2), and Cho and Moreno's (2024) (5) which are all (somewhat more complicated) variations of (104) from McClung (2021).

or $x_t = \lambda_j^{-1} E_t[x_{t+1}]$ where $x_t = b_t - \lambda_i b_{t-1}$. If λ_j is outside and λ_i inside the unit circle, then $x_t = 0$ is the unique stationary solution for x_t following Blanchard (1979) of the forward looking equation and $b_t = \lambda_i b_{t-1}$ for b_t. Inspection shows that one λ, say λ_1, is ϕ_π and the other, thus λ_2, is $\beta^{-1} - \phi_d$. If $|\phi_\pi| > 1$ and $|\beta^{-1} - \phi_d| > 1$ we have two unstable roots and a stationary solution does not exists, if $|\phi_\pi| < 1$ and $|\beta^{-1} - \phi_d| < 1$ we have two stable roots and two stationary solutions exist and thus the solution is indeterminate. The case of $|\phi_\pi| > 1$ and $|\beta^{-1} - \phi_d| < 1$, and hence $\lambda_j = \lambda_1$ and $\lambda_i = \lambda_2$, is the case of active monetary and passive fiscal policy - the Taylor principle. Leeper (1991) introduced an alternative possibility, namely the case of $|\phi_\pi| < 1$ and $|\beta^{-1} - \phi_d| > 1$, and hence $\lambda_j = \lambda_2$ and $\lambda_i = \lambda_1$, the case of active fiscal and passive monetary policy.

This determinacy result reemerges in the frictionless model and is essentially immediate that this carries over to the models with imperfect information as in the follows, where we have re-adopted the assumption throughout the paper that $\phi_\pi > 0$

Theorem 12 (Informational Inertia and Leeper's (1991) Equilibria under Monetary and Fiscal Policies)**.** *A model of informational inertia, given by (28), (74), or (78) on the supply side; (2) on the demand side; the Taylor rule (1) for monetary policy; (104) for debt dynamics; and (105) for the fiscal rule has a unique, stable equilibrium if and only if*

- either (a) $\phi_\pi > 1$ *and* $|\beta^{-1} - \phi_d| < 1$ or (b) $\phi_\pi < 1$ *and* $|\beta^{-1} - \phi_d| > 1$

abstracting from non-generic collinearity in the resolution of prediction errors.

Our theorem 12 emphasizes that the boundary between active and passive monetary policy is given by the unadulterated Taylor principle, unless one accepts a non-vertical long run Phillips curve. The Taylor principle is, as in our pure monetary analysis above, again weakened by the sticky price assumption. The result is contained in Woodford's (2003) standard reference, but frequently just briefly acknowledged, e.g., Cho and Moreno (2024, p. 5) or McClung (2021, p. 100). We repeat in the following to emphasize the difference to our conservative bounds

Theorem 13 (Sticky Prices and Leeper's (1991) Equilibria under Monetary and Fiscal Policies)**.** *The sticky price model, given by the sticky price Phillips curve (8) on the supply side; (2) on the demand side; the Taylor rule (1) for monetary policy; (104) for debt dynamics; and (105) for the fiscal rule has a unique, stable equilibrium if and only if*

- either (a) $\frac{1-\beta}{\kappa}\phi_y + \phi_\pi > 1$ *and* $|\beta^{-1} - \phi_d| < 1$ or (b) $\frac{1-\beta}{\kappa}\phi_y + \phi_\pi < 1$ *and* $|\beta^{-1} - \phi_d| > 1$

Proof. See Woodford (2003, Proposition 4.11). □

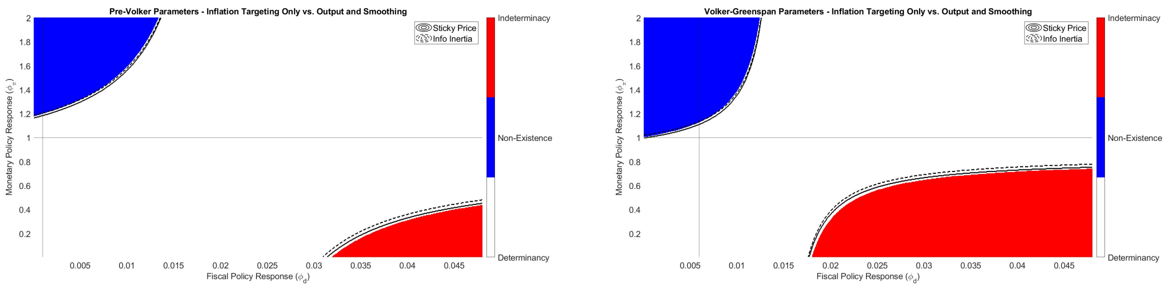
McClung (2021) and Cho and Moreno (2024) also combine the fiscal-monetary interaction of this subsection with the regime switching approach of Davig and Leeper (2007) addressed in the previous subsection. While this provides significant insights into forward guidance and exits from the effective lower bound of nominal interest rates, the determinacy bounds under regime switching with fiscal/monetary interaction and the different supply sides follow the same pattern as above, with more conservative bounds following from the strict Taylor principle under long run verticality of the supply curve. As Woodford (2003, p. 685) points out in his proof of his Proposition 4.11 that we use above, the dynamics of debt can be separated block triangularly from the dynamics of the remainder of the model, i.e., the monetary block. This is what enabled the active/passive threshold of fiscal policy to be independent of (and hence identical across) supply curves above and the active/passive thresholds of monetary policy to coincide with those from the pure monetary determinacy analysis in previous sections here.

Canzoneri, Cumby, and Diba (2010) and Canzoneri, Cumby, Diba, and López-Salido (2011) (henceforth CCD10/CCDLS11) provide exactly such an interdependence with their model of liquid government bonds. They add to the analyses above the assumption that households need liquidity to facilitate transactions and that government bonds are liquid but imperfect substitutes for money in this facilitation. The key insight for our analysis comes from their earlier paper, Canzoneri and Diba (2005), which analyzes the impact of this transactions role of bonds for determinacy in a model with fully flexible prices. In contrast to the studies above, the dynamics of inflation and debt are not decoupled but jointly determine transactions - fiscal policy determines total transactions balances, monetary policy converts these into effective transactions balances - hence the block triangularity of Woodford (2003) from above no longer holds. That is, the separation between indeterminacy, nonexistence, and determinacy does not follow the active/passive fiscal/monetary scheme of Leeper (1991). Canzoneri and Diba (2005) show that the critical value for ϕ_π in determining the threshold for determinacy with passive fiscal policy depends on the specification and numerical value in the feedback rule of fiscal policy - the transactions role reduces the critical value for ϕ_π below one and a more vigorous fiscal policy increases the threshold back towards one.

As the exercise is now inherently numerical, we return to the model of CCD10/CCDLS11 that includes a Phillips curve. They take up the empirical assessment of determinacy by addressing the analysis of Lubik and Schorfheide (2004) that finds indeterminacy in the pre-Volker and determinacy in the Volker-Greenspan era. In contrast, CCD10/CCDLS11 find that both eras were associated with a determinate solution despite the former period having had passive monetary policy. We reproduce their analysis of determinacy and supplement their sticky-price with a long-run vertical Phillips curve - apart from the supply specification, we adopt their baseline calibration for comparability. The analysis begins with assuming the interest rate responds only to inflation and then examine implications of responses to output and the lagged rate, so conforming to (101), as

$$R_t = \rho_R R_{t-1} + (1-\rho_R)\left[\phi_\pi \pi_t + \phi_y y_t\right] \tag{108}$$

and the fiscal rule (105) above. Beginning with the case $\rho_R = \phi_y = 0$, monetary policy only responds to inflation so the specification of the Phillips curve is irrelevant. This can be seen in figure 3 with the determinacy/existence regions identical for both specification of the Phillips curve - for the $\rho_R = \phi_y = 0$ case, these are the filled regions. The effect of the coupling of fiscal and monetary policy via the transactions role of bonds for determinacy can be clearly seen - the standard active/passive fiscal/monetary policy regions are divided by at the Taylor principle threshold $\phi_\pi = 1$ for monetary and $\phi_d = 1/\beta$ for fiscal policy, that is, the FTPL analyses above would fill the upper left quadrant blue and the lower right red. The red and lower right quadrant are the focus of our analysis and consistent with Canzoneri and Diba (2005), monetary policy is not constrained by the Taylor principle and can implement a unique equilibrium with a ϕ_π less than one, the critical response increases in ϕ_d as fiscal policy responds more vigorously to government debt.



(A) Pre Volcker Parameters

(B) Volcker Greenspan Parameters

FIGURE 3. Stable/indeterminate/nonexistence regions, and empirical ranges

In both figures, the gray boxes are the ranges of ϕ_d and ϕ_π reported by CCD10/CCDLS11.

Shaded: nonexistence/indeterminate regions with monetary policy responding only to inflation (identical for both pricing fictions). Curves: bounds under the general specification.

Turning to the more general case, we set $\rho_R = 0.6$ and $\phi_y = 0.15$ following the empirical analysis of CCD10/CCDLS11 and re-examine the determinacy and existence regions based on ϕ_d and ϕ_π. The critical values are traced out in figure 3. In comparison to the filled regions for the $\rho_R = \phi_y = 0$ case, the lower bounds for the response of monetary policy ϕ_π now differ depending on the Phillips curve. In the lower right quadrant, the solid line (sticky prices) lies below the dashed line (informational inertia). This is consistent with our analysis above: a response to output by monetary policy enlarges the determinacy region under sticky prices as its Phillips curve has a stable, long-run tradeoff.

9. CONCLUSION

We have derived determinacy bounds on monetary policy under a vertical long run Phillips curve. In contrast to the standard sticky price result, we find that only the response to current inflation matters for determinacy. If the long run Phillips curve is vertical, no amount of output gap targeting, forward or backward-looking inflation targeting can substitute for a more than one-for-one response to current inflation directly. Policy makers with a concern for robustness and reluctance to posit a long run tradeoff between output and inflation ought to prefer our bounds and heed the Taylor principle.

We have shown this with a generic model of finite informational inertia and two specific models, the sticky information model of Mankiw and Reis (2002) and the imperfect common knowledge model of Nimark (2008), the former by formulating it as a recursion in the frequency domain and the latter by identifying a time domain recursion by defining a higher order expectation operator. The transformations of the models separate the long-run dynamic relationships between variables that establish the determinacy properties from sequences of forecasting errors analogously to the separation of the homogenous component of a difference equation relation from the particular solution. We complement the two models from the literature with our generic model of information rigidity that is specified only as imposing the natural rate hypothesis at some horizon, confirming the generality of our results. We show that the conclusion is robust to more general specifications of the Taylor rule, including regime shifting, and transfers to active/passive fiscal/monetary policy in FTPL analyses.

In sum, the paper has shown that determinacy bounds on monetary policy to deliver a unique, stationary equilibrium is a long run phenomenon. Models that share a long-run vertical Phillips curve also share determinacy bounds. Our bounds are more restrictive than existing bounds from sticky price models that enable these bounds to be widened through the models' stable long-run tradeoff between inflation and output. We reject these wider bounds and advocate a literal implementation of the Taylor principle.

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Appendix A. Proof of Theorem 2

While the proof can be found elsewhere, Woodford (2003), Galí (2008), Bullard and Mitra (2002), or Lubik and Marzo (2007), it is instructive to repeat it here to make the transition to establishing determinacy for the sticky information model more straightforward. Combining (1), (2), and (8) gives

$$\begin{bmatrix} -\beta & 0 \\ \sigma & 1 \end{bmatrix} \begin{bmatrix} E_t\pi_{t+1} \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} -1 & \kappa \\ \sigma\phi_\pi & 1+\sigma\phi_y \end{bmatrix} \begin{bmatrix} \pi_t \\ y_t \end{bmatrix} \tag{A.1}$$

which for $\beta \neq 0$ can be inverted to yield

$$\begin{bmatrix} E_t\pi_{t+1} \\ E_t y_{t+1} \end{bmatrix} = A \begin{bmatrix} \pi_t \\ y_t \end{bmatrix}, \text{ where } A = \begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ \sigma(\phi_\pi - \frac{1}{\beta}) & 1+\frac{\sigma}{\beta}\kappa + \sigma\phi_y \end{bmatrix} \tag{A.2}$$

If both eigenvalues of A lie outside the unit circle, then solving forward gives

$$\begin{bmatrix} \pi_t \\ y_t \end{bmatrix} = \lim_{j\to\infty} A^{-j} \begin{bmatrix} E_t\pi_{t+j} \\ E_t y_{t+j} \end{bmatrix} \tag{A.3}$$

unique, bounded solution for π_t and y_t following Blanchard (1979).

The Schur-Cohn criteria, (see LaSalle, 1986, p.28), to ascertain whether both eigenvalues indeed do lie outside the unit circle are $|\det(A)| > 1$ and $|\mathrm{tr}(A)| < 1 + \det(A)$. As

$$\det(A) = \frac{1}{\beta}(1+\sigma\phi_y + \kappa\sigma\phi_\pi) > 1 \text{ and } \mathrm{tr}(A) = \frac{1}{\beta} + \frac{\sigma\kappa}{\beta} + 1 + \sigma\phi_y > 1 \tag{A.4}$$

the condition $|\det(A)| > 1$ necessarily holds and $|\mathrm{tr}(A)| < 1 + \det(A)$ holds if

$$1 < \frac{1-\beta}{\kappa}\phi_y + \phi_\pi \tag{A.5}$$

Appendix B. Proof of Theorem 2

Combining the IS curve (2) and the Taylor rule (1)

$$\left(1+\sigma\phi_y\right) y_t = E_t y_{t+1} - \sigma\phi_\pi\pi_t + \sigma E_t\pi_{t+1} \tag{B.6}$$

inserting the Phillips curve (10)

$$\left(1+\sigma\phi_y\right)\alpha\left(\pi_t - E_{t-1}\pi_t\right) = \alpha E_t\left[\pi_{t+1} - E_t\pi_{t+1}\right] - \sigma\phi_\pi\pi_t + \sigma E_t\pi_{t+1} \tag{B.7}$$

$$= -\sigma\phi_\pi\pi_t + \sigma E_t\pi_{t+1} \tag{B.8}$$

and now taking time $t-1$ expectations and recalling the law of iterated expectations

$$0 = -\sigma\phi_\pi E_{t-1}\pi_t + \sigma E_{t-1}E_t\pi_{t+1} \tag{B.9}$$

or the time $t-1$ expectations version of (6)

$$\phi_\pi\pi_{t|t-1} = E_{t-1}\pi_{t+1|t} \tag{B.10}$$

where $\pi_{t|t-1} \equiv E_{t-1}\pi_t$ and solving forward, Blanchard (1979)

$$\pi_{t|t-1} = \lim_{j\to\infty}\frac{1}{\phi_\pi^j}E_{t-1}\pi_{t+j|t+j-1} = \lim_{j\to\infty}\frac{1}{\phi_\pi^j}E_{t-1}\pi_{t+j} \tag{B.11}$$

delivers a unique, bounded solution for $\pi_{t|t-1}$ if and only if $1 < \phi_\pi$. This determines the value for $\pi_{t|t-1}$ (and hence $\pi_{t+1|t}$ by the time invariance of the problem) so

$$\pi_t = \frac{1}{\left(1+\sigma\phi_y\right)\alpha + \sigma\phi_\pi}\left(\sigma E_t\pi_{t+1} + \left(1+\sigma\phi_y\right)\alpha E_{t-1}\pi_t\right) \tag{B.12}$$

and y_t then from (10) and R_t from (1).

Appendix C. Proof of Proposition 1

(1) The evaluation inside the unit disk of a function defined on the unit disk by the integral of its value on the boundary (unit circle), can be found in Garnett (1981, section 1.3), Theorem 11.8 in Rudin (1987), Theorem 3.1 Duren (1970) and Ahlfors (1979, pp. 166).

(2) The kernel P_λ is nonnegative, even, has unit mass, and forms an approximate identity: $P_\lambda \to \delta_0$ as $\lambda \to 1$ and $P_\lambda \to 1$ as $\lambda \to 0$ (see Items (i)–(vii) Garnett (1981, p. 13), Rudin (1987, p. 233), and Duren (1970, section 1.1)). This gives the basis for the averaging and filtering interpretations.

Take $x \in H^2$ with Taylor (impulse) series $x(z) = \sum_{j\geq 0} x_j z^j$ and boundary function $h(\theta) := x(e^{i\theta}) \in L^2[-\pi,\pi]$. For $0<\lambda<1$ the Poisson kernel on the circle can also be written as

$$P_\lambda(\phi) = \frac{1-\lambda^2}{1-2\lambda\cos\phi+\lambda^2} = \sum_{n\in\mathbb{Z}} \lambda^{|n|}e^{in\phi} = 1 + 2\sum_{n\geq 1}\lambda^n\cos(n\phi), \tag{C.13}$$

and satisfies $P_\lambda \geq 0$ and $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_\lambda(\phi)\,d\phi = 1$ from above, giving the Fourier series.

Write the boundary Fourier series $h(\theta) = \sum_{n\in\mathbb{Z}}\widehat{h}(n)e^{in\theta}$. Since $x \in H^2$, h has only nonnegative modes and $\widehat{h}(n) = x_n$ for $n \geq 0$, while $\widehat{h}(n) = 0$ for $n<0$; from above. Convolution with P_λ acts diagonally on Fourier modes:

$$(P_\lambda * h)(\omega) = \sum_{n\in\mathbb{Z}} \lambda^{|n|}\widehat{h}(n)e^{in\omega} = \sum_{j\geq 0}\lambda^j x_j e^{ij\omega} = x(\lambda e^{i\omega}) \tag{C.14}$$

which shows that the nth harmonic is multiplied by $\lambda^{|n|}$, giving the exponential damping of higher harmonics. Because $\widehat{h}(j) = x_j$ for $j \geq 0$, the same multipliers appear on the impulse coefficients, giving the relationship

$$x_j \;\mapsto\; \lambda^j x_j \tag{C.15}$$

for $< x(\lambda z)$, giving the exponential damping by lag. Finally, since $P_\lambda \geq 0$ and integrates to 1, $(P_\lambda * h)(\omega)$ is a weighted average of boundary values, hence a low–pass average; as $\lambda \uparrow 1$, $P_\lambda * h \to h$, while as $\lambda \downarrow 0$, $P_\lambda * h \to \frac{1}{2\pi}\int h$ from Garnett (1981, section I.3).

The interpretation of the Poisson kernel as a filter follows by inspection and comparison with Brillinger (2001, section 3.3) on convergence factors bandwidth tables and both Oppenheim and Schafer (2010, section 10.6) and Priestley (1981, section 6.2.3) on windowing. Items (i)–(vii) Garnett (1981, p. 13) gives the identity with the radial limits: these are exactly the Poisson-kernel properties and boundary convergence statements formulas (3.1)–(3.2) Garnett (1981, section I.3)

(3) This follows the standard covariance formula for causal linear filters; see, e.g., Brillinger (2001, section 2.2) or Priestley (1981, section 6.2.3). The autocovariance of the process $x_t^{(\lambda)} = x(\lambda L)\varepsilon_t$, $\gamma_\lambda(m)$ is

$$\gamma_\lambda(m) = \sigma^2\sum_{j\geq 0}\lambda^j x_j\,\lambda^{j+|m|}x_{j+|m|} = \lambda^{|m|}\sigma^2\sum_{j\geq 0}\lambda^{2j}x_j x_{j+|m|}. \tag{C.16}$$

So for squared summable x_j

$$|\gamma_\lambda(m)| \leq \lambda^{|m|}\sigma^2 \sum_{j\geq 0} \lambda^{2j}|x_j|\,|x_{j+|m|}| \leq \lambda^{|m|}\sigma^2\,\|x\|_2^2, \tag{C.17}$$

By Cauchy–Schwarz,

$$\sum_{j\geq 0} \lambda^{2j}|x_j|\,|x_{j+|m|}| \leq \left(\sum_{j\geq 0} \lambda^{2j} x_j^2\right)^{1/2} \left(\sum_{j\geq 0} \lambda^{2j} x_{j+|m|}^2\right)^{1/2} = \|x^{(\lambda)}\|_2^2 \leq \|x\|_2^2, \tag{C.18}$$

which yields the bound. Thus $|\gamma_\lambda(m)| \leq C\,\lambda^{|m|}$ with $C = \sigma^2\|x\|_2^2$, the exponential memory shortening.

(4) The log spectrum convolution follows directly from Theorem 4.6 in Garnett (1981, p. 67).

Appendix D. Proof of Proposition 2

If $f(z) = \sum_{k\geq 0} a_k z^k \in H^2$, then $\|C_\lambda f\|_{H^2}^2 = \sum_{k\geq 0} |a_k|^2\lambda^{2k} \leq \|f\|_{H^2}^2$, so C_λ is a contraction on H^2, see Theorem 17.12 in Rudin (1987, pp. 341-342). Since $G \in H^\infty$ with $\|G\|_\infty \leq \frac{\lambda}{|\xi|(1-\lambda)}$, and multiplication by a bounded analytic function acts boundedly on H^2; the multiplier M_G is bounded, see Garnett (1981, section II.3), as H^∞ acts boundedly on H^2, and $\{z^n\}_{n\geq 0}$ is an orthonormal basis, see Rudin (1987, pp. 88–92). Moreover $\|\lambda C_\lambda\| \leq \lambda < 1$, hence the inverse exists as a von Neumann series in a Banach algebra and is given by the operator norm convergent series

$$(I - \lambda C_\lambda)^{-1} = \sum_{m=0}^{\infty} (\lambda C_\lambda)^m = \sum_{m=0}^{\infty} \lambda^m\, C_{\lambda^m}, \tag{D.19}$$

with $\left|(I-\lambda C_\lambda)^{-1}\right| \leq (1-\lambda)^{-1}$, see, e.g., Theorem 18.3 Rudin (1987, pp.357) or Theorem 10.7 Rudin (1991, pp. 249-250). Therefore N is bounded, so $\pi \in H^2$ means $y = N[\pi] \in H^2$.

For the bounded left inverse, assume to the contrary that there exists a bounded left inverse. That is, there is a bounded operator B with $BN = I$ on H^2 (a bounded left inverse). Then N can be bounded below using the submultiplicativity of the operator norm, see Rudin (1991, sections 4.17 and 10.1),

$$\|f\| = \|BNf\| \leq \|B\|\,\|Nf\| \quad \Rightarrow \quad \|Nf\| \geq \|B\|^{-1}\|f\|. \tag{D.20}$$

Now defining $e_n(z) := z^n$ and $\{e_n\}_{n\geq 0}$ is an orthonormal basis of H^2, Garnett (1981, pp. 59-60). Because $C_\lambda e_n = \lambda^n e_n$, it follows that

$$\|Ne_n\| = \|(I-\lambda C_\lambda)^{-1} M_G C_\lambda e_n\| \leq \|(I-\lambda C_\lambda)^{-1}\|\,\|M_G\|\,\|C_\lambda e_n\| \leq \frac{\|G\|_\infty}{1-\lambda}\,\lambda^n \tag{D.21}$$

Since $0 < \lambda < 1$, the right–hand side goes to 0 as n goes to ∞, contradicting bound from below. Hence no bounded B with $BN = I$ exists, and so N has no bounded left inverse.

Appendix E. Proof of Proposition 3

Inserting, (17), the moving average representation with respect to a demand shock for y_t and π_t, into the sticky information Phillips curve (28) and matching terms on ϵ_{t-j} gives

$$\pi_k = \frac{1-\lambda}{\lambda}\xi y_k + (1-\lambda)\sum_{j=0}^{i-1} \lambda^j \left[\pi_k + \xi(y_k - y_{k-1})\right] \tag{E.22}$$

or collecting terms

$$\lambda^{k+1}\pi_k = \left(1-\lambda^{k+1}\right)\xi y_k - \xi\lambda\left(1-\lambda^k\right) y_{k-1} \tag{E.23}$$

which can also be written as

$$y_k = \frac{\lambda^{k+1}}{1-\lambda^{k+1}} \frac{1}{\xi} \left(\sum_{j=0}^{k} \pi_j \right) \tag{E.24}$$

Clearly for any bounded sequence of π_k, that is $\lim_{k\to\infty} |\sum_{j=0}^{k} \pi_j| < \infty$, y_k converges to zero. For unbounded sequences of π_k such that $\pi_k < \alpha^j \pi_{k-j}$ and $|\alpha\lambda| < 1$ (that is, $\alpha < 1/\lambda$), y_k also converges to zero.

and the slope of the Phillips curve is

$$1/LRS|_{\text{Sticky Information}} = \lim_{k\to\infty} \frac{\partial y_{t+k}}{\partial \epsilon_t} \left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t} \right]^{-1} \tag{E.25}$$

$$= \lim_{k\to\infty} \left(\frac{\lambda^{k+1}}{1-\lambda^{k+1}} \frac{1}{\xi} \left(\sum_{j=0}^{k} \frac{\partial \pi_{t+j}}{\partial \epsilon_t} \right) \right) \left[\frac{\partial \pi_{t+k}}{\partial \epsilon_t} \right]^{-1} \tag{E.26}$$

$$LRS|_{\text{Sticky Information}} = \lim_{k\to\infty} \frac{\partial \pi_{t+k}}{\partial \epsilon_t} \left[\frac{\partial y_{t+k}}{\partial \epsilon_t} \right]^{-1} \tag{E.27}$$

$$= \lim_{k\to\infty} \frac{1-\lambda^{k+1}}{\lambda^{k+1}} \xi - \lim_{k\to\infty} \frac{1-\lambda^{k}}{\lambda^{k}} \xi \frac{\partial y_{t+k-1}}{\partial \epsilon_t} \left[\frac{\partial y_{t+k}}{\partial \epsilon_t} \right]^{-1} \tag{E.28}$$

$$= - \lim_{k\to\infty} \frac{1-\lambda^{k}}{\lambda^{k}} \xi \frac{\partial y_{t+k-1}}{\partial \epsilon_t} \left[\frac{\partial y_{t+k}}{\partial \epsilon_t} \right]^{-1} \tag{E.29}$$

APPENDIX F. PROOF OF THEOREM 4

At $z = 0$, define $y(0) = y_0$, $\pi(0) = \pi_0$, the sticky information Phillips curve (35) is determined by

$$\xi \frac{1-\lambda}{\lambda} y_0 = \pi_0 \tag{F.30}$$

which yields one initial condition: inflation at $z = 0$ is a constant share of output increasing in the share of firms that received an information update in the initial period $1-\lambda$. The remaining condition at $z = 0$ must follow from the system given by the Phillips curve (35)

$$\frac{\xi}{\lambda} y(z) = z\xi y(z) + \pi(\lambda z) + \xi(1-\lambda z) y(\lambda z) \tag{F.31}$$

and the IS curve equation with the interest rate rule (1)

$$(1+\sigma\phi_y) z y(z) + \sigma\phi_\pi z \pi(z) = y(z) - y_0 + \sigma(\pi(z) - \pi_0) \tag{F.32}$$

The matrix system is determined by (F.30), (85) and (F.32) as

$$\begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} = \begin{bmatrix} \phi_\pi & \frac{1+\sigma\phi_y-\lambda}{\sigma} \\ 0 & \lambda \end{bmatrix} z \begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} + \begin{bmatrix} \frac{1-\lambda}{\lambda}\xi + \frac{1}{\sigma} \\ 0 \end{bmatrix} y_0 + \begin{bmatrix} -\frac{\lambda}{\sigma\xi} & -\frac{\lambda}{\sigma}(1-\lambda z) \\ \frac{\lambda}{\xi} & \lambda(1-\lambda z) \end{bmatrix} \begin{bmatrix} \pi(\lambda z) \\ y(\lambda z) \end{bmatrix} \tag{F.33}$$

If $[\pi(\lambda z), y(\lambda z)]'$ are analytic functions $\forall\ |z| < 1$, then $[\pi(z), y(z)]'$ are analytic functions $\forall\ |z| < \frac{1}{\lambda}$ and as $0 < \lambda < 1$ certainly for $|z| < 1 < \frac{1}{\lambda}$. Similarly to (C.38) the system of equations can be expressed as

$$(I_2 - zA) \begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} = \begin{bmatrix} \frac{1-\lambda}{\lambda}\xi \\ 0 \end{bmatrix} y_0 + \begin{bmatrix} -\frac{\lambda}{\sigma\xi} & -\frac{\lambda}{\sigma}(1-\lambda z) \\ \frac{\lambda}{\xi} & \lambda(1-\lambda z) \end{bmatrix} \begin{bmatrix} \pi(\lambda z) \\ y(\lambda z) \end{bmatrix} \tag{F.34}$$

where $A = \begin{bmatrix} \phi_\pi & \frac{1+\sigma\phi_y-\lambda}{\sigma} \\ 0 & \lambda \end{bmatrix}$. The eigenvalues of matrix A are $\rho_1 = \phi_\pi, \rho_2 = \lambda$ which can be factored as $A = V\Lambda V^{-1}$ where Λ is the matrix of eigenvalues, giving us

$$\begin{bmatrix} w(z) \\ u(z) \end{bmatrix} = V^{-1} \begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} \tag{F.35}$$

where $V = \begin{bmatrix} 1 & \frac{1+\sigma\phi_y-\lambda}{\sigma(\lambda-\phi_\pi)} \\ 0 & 1 \end{bmatrix}$ and $V^{-1} = \begin{bmatrix} 1 & -\frac{1+\sigma\phi_y-\lambda}{\sigma(\lambda-\phi_\pi)} \cdot \\ 0 & 1 \end{bmatrix}$.

The system of equations can be diagonalized in $w(z)$ and $u(z)$ as

$$(I_2 - z\Lambda)\begin{bmatrix} w(z) \\ u(z) \end{bmatrix} = \begin{bmatrix} \frac{1-\lambda}{\lambda}\xi + \frac{1}{\sigma} \\ 0 \end{bmatrix} u_0 + \begin{bmatrix} -\frac{\lambda}{\xi}(\frac{1}{\sigma}+v_{12}) & -\frac{\lambda}{\xi}(\frac{1}{\sigma}+v_{12})(\xi+v_{12})(1-\frac{\xi\lambda}{\xi+v_{12}}z) \\ \frac{\lambda}{\xi} & \frac{\lambda}{\xi}(\frac{1}{\sigma}+v_{12})(1-\frac{\xi\lambda}{\xi+v_{12}}z) \end{bmatrix} \begin{bmatrix} w(\lambda z) \\ u(\lambda z) \end{bmatrix} \tag{F.36}$$

The first equation is given by

$$(1-z\phi_\pi)w(z) = \left(\frac{1-\lambda}{\lambda}\xi + \frac{1}{\sigma}\right)u_0 - \frac{\lambda}{\xi}\left(\frac{1}{\sigma}+u_{12}\right)w(\lambda z) - \frac{\lambda}{\xi}\left(\frac{1}{\sigma}+v_{12}\right)(\xi+v_{12})\left(1-\frac{\lambda\xi}{\xi+v_{12}}z\right)u(\lambda z). \tag{F.37}$$

Iff $\phi_\pi > 1$ there is a removable singularity, which provides the additional initial condition

$$\lim_{z\to\frac{1}{\phi_\pi}} (1-z\phi_\pi)w(z) = 0 \tag{F.38}$$

which uniquely determines the missing initial condition u_0

$$\left(\frac{1-\lambda}{\lambda}\xi + \frac{1}{\sigma}\right)u_0 = \frac{\lambda}{\xi}\left(\frac{1}{\sigma}+v_{12}\right)\left(w\left(\frac{\lambda}{\phi_\pi}\right) + (\xi+v_{12})\left(1-\frac{\lambda\xi}{\xi+v_{12}}\frac{1}{\phi_\pi}\right)u\left(\frac{\lambda}{\phi_\pi}\right)\right) \tag{F.39}$$

from which together with (F.35) and (F.30) we can therefore deduce $\pi_0 = 0$ and $y_0 = 0$.[32]

Appendix G. Recursive Representation of the Imperfect Common Knowledge Phillips Curve

The main text derived the recursive representation using the average higher order expectations operator we defined as $H_s x_t \equiv \int E[x_t|\mathscr{I}_s(j)]dj$. Here we derive the recursive representation using an alternative

[32] Note that (F.39) determines u_0 only implicitly, i.e., in dependence of $u\left(\frac{\lambda}{\phi_\pi}\right)$ and $w\left(\frac{\lambda}{\phi_\pi}\right)$. Hence for this homogenous solution where the zero solution is always a solution, uniqueness implies the solution is the zero solution, see footnote 9. As stated above, ascertaining that the equilibrium is unique is different than calculating the equilibrium itself and we proceeded without loss of generality with respect to determinacy in the absence of exogenous shocks. When confronted with exogenous shocks, u_0 would have to be jointly solved with $u\left(\frac{\lambda}{\phi_\pi}\right)$ and $w\left(\frac{\lambda}{\phi_\pi}\right)$ via the system of equations

$$\begin{bmatrix} w(z) \\ u(z) \end{bmatrix} = \begin{bmatrix} \frac{\frac{1-\lambda}{\lambda}\xi+\frac{1}{\sigma}}{1-\phi_\pi z} \\ 0 \end{bmatrix} u_0 + \frac{\lambda}{\xi}\frac{1}{1-\phi_\pi z}\begin{bmatrix} -(\frac{1}{\sigma}+v_{12}) & -(\frac{1}{\sigma}+v_{12})(\xi+v_{12})(1-\frac{\lambda\xi}{\xi+v_{12}}z) \\ 1 & (\xi+v_{12})(1-\frac{\lambda\xi}{\xi+v_{12}}z) \end{bmatrix} \begin{bmatrix} w(\lambda z) \\ u(\lambda z) \end{bmatrix} \tag{F.40}$$

That is, while we can analytically solve for determinacy conditions in the sticky information model with forward looking demand (2), this approach does not let us analytically solve for, say, impulse responses to inhomogenous shocks.

route. Begin with the Phillips curve (47)

$$\pi_t = (1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k\, mc_{t|t}^{(k)} + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k\,\pi_{t+1|t}^{(k+1)} \tag{G.41}$$

Calculate the average higher order expectation of π_t, $\pi_{t|s}^{(1)} \equiv \int E\left[\pi_t|\mathscr{I}_s(j)\right]dj$

$$\int E\left[\pi_t|\mathscr{I}_s(j)\right]dj = \int E\left[(1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k\, mc_{t|t}^{(k)} + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k\,\pi_{t+1|t}^{(k+1)}\middle|\mathscr{I}_s(j)\right]dj \tag{G.42}$$

$$\pi_{t|s}^{(1)} = (1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k\int E\left[mc_{t|t}^{(k)}\middle|\mathscr{I}_s(j)\right]dj \tag{G.43}$$

$$+\beta\theta\sum_{k=0}^{\infty}(1-\theta)^k\int E\left[pi_{t+1|t}^{(k+1)}\middle|\mathscr{I}_s(j)\right]dj \tag{G.44}$$

$$\pi_{t|s}^{(1)} = (1-\theta)\left(1-\beta\theta\right)\sum_{k=0}^{\infty}(1-\theta)^k\, mc_{t|t}^{(k+1)} + \beta\theta\sum_{k=0}^{\infty}(1-\theta)^k\,\pi_{t+1|t}^{(k+2)} \tag{G.45}$$

multiply with $(1-\theta)$ and compare with (G.41)

$$(1-\theta)\pi_{t|s}^{(1)} = (1-\theta)\left(1-\beta\theta\right)\sum_{k=1}^{\infty}(1-\theta)^k\, mc_{t|t}^{(k)} + \beta\theta\sum_{k=1}^{\infty}(1-\theta)^k\,\pi_{t+1|t}^{(k+1)} \tag{G.46}$$

$$= \pi_t - (1-\theta)\left(1-\beta\theta\right)mc_t - \beta\theta\pi_{t+1|t}^{(1)} \tag{G.47}$$

or

$$\pi_t - (1-\theta)\pi_{t|s}^{(1)} = (1-\theta)\left(1-\beta\theta\right)mc_t + \beta\theta\pi_{t+1|t}^{(1)} \tag{G.48}$$

which gives (52) in the main text.

Appendix H. Proof of Theorem 5

Following the proof of theorem 2 we can combine (2), (55), with the Taylor rule (1) as

$$\begin{bmatrix} E_t\pi_{t+1} \\ E_t y_{t+1}\end{bmatrix} = A\begin{bmatrix}\pi_t \\ y_t\end{bmatrix} - \begin{bmatrix}\frac{1}{\beta} \\ 0\end{bmatrix}\zeta_t \tag{H.49}$$

where $A = \begin{bmatrix}\frac{1}{\beta} & -\frac{\kappa}{\beta} \\ \sigma(\phi_\pi - \frac{1}{\beta}) & 1+\frac{\sigma}{\beta}\kappa + \sigma\phi_y\end{bmatrix}$ and $\zeta_t = -\frac{1-\theta}{\theta}\int(\pi_t - E\left[\pi_t|\mathscr{I}_s(j)\right])dj - \beta\int(E_t\pi_{t+1} - E\left[\pi_t|\mathscr{I}_s(j)\right])dj$ is an increment process of forecast/prediction errors. Determinacy of the inhomogenous system requires determinacy of the homogenous system, which like in theorem 2 requires both eigenvalues of A be outside the unit circle, or $1-\frac{1-\beta}{\kappa}\phi_y < \phi_\pi$ restricting ourselves to positive coefficients.

Alternatively, we can appeal to frequency domain methods as in theorem 1 in the online appendix. Applying the z-transform to the above gives

$$(I_2 - zA)\begin{bmatrix}\pi(z) \\ y(z)\end{bmatrix} = \begin{bmatrix}\pi_0 \\ y_0\end{bmatrix} - \begin{bmatrix}\frac{1}{\beta} \\ 0\end{bmatrix}\zeta(z) \tag{H.50}$$

we require $\pi(z)$, $y(z)$, and $\zeta(z)$ to be analytic on the unit disk and (H.50), along with the underlying prediction/forecast error definition of $\zeta(z)$, will fail to provide a unique set of restrictions unless we can pin down π_0 and y_0. Precisely when both eigenvalues of A are outside the unit circle, can we appeal as in theorem 1 in the online appendix to Cauchy's residue theorem to provide a unique set of restrictions on π_0 and y_0 such that

$$\begin{bmatrix} \pi(z) \\ y(z) \end{bmatrix} = (I_2 - zA)^{-1}\left(\begin{bmatrix} \pi_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} \frac{1}{\beta} \\ 0 \end{bmatrix} z\zeta(z)\right) \tag{H.51}$$

continues to be well defined via analytic continuation over these two values of z inside the unit circle.

Appendix I. Proof of Theorem 6

We can let $\kappa \to \infty$ in theorem 5.

Or we can proceed in the time domain following the proof of theorem 1 we can combine (2), (74), with the Taylor rule (1) as

$$\phi_\pi \pi_t = E_t \pi_{t+1} - \frac{1+\sigma\phi_y}{\sigma}\tilde{\zeta}_t + \frac{1}{\sigma} E_t \tilde{\zeta}_{t+1} \tag{I.52}$$

where $\tilde{\zeta}_t = \frac{1}{\Theta}\int (\pi_t - E\left[\pi_t | \mathscr{I}_s(j)\right]) dj$. Solving forward, Blanchard (1979)

$$\pi_t = \lim_{j\to\infty} \frac{1}{\phi_\pi^j} E_t \pi_{t+j} - \sum_{j=0}^{\infty} \frac{1}{\phi_\pi^j}\left(\frac{1+\sigma\phi_y}{\sigma} E_t \tilde{\zeta}_{t+j} - \frac{1}{\sigma} E_t \tilde{\zeta}_{t+j+1}\right) \tag{I.53}$$

delivers a unique, bounded solution for π_t if and only if $1 < \phi_\pi$. That is, only if $1 < \phi_\pi$ does the sunspot term $\lim_{j\to\infty} \frac{1}{\phi_\pi^j} E_t \pi_{t+j}$ disappear.

Alternatively, we can appeal to frequency domain methods as in theorem 1 in the online appendix. Applying the z-transform to the above gives

$$\left(1 - z\phi_\pi\right)\pi(z) = \pi_0 - \frac{1+\sigma\phi_y}{\sigma} z\tilde{\zeta}(z) + \frac{1}{\sigma}\left(\tilde{\zeta}(z) - \tilde{\zeta}(0)\right) \tag{I.54}$$

We require $\pi(z)$ and $\tilde{\zeta}(z)$ to be analytic on the unit disk and the foregoing, along with the underlying prediction/forecast error definition of $\tilde{\zeta}(z)$, will fail to provide a unique set of restrictions unless we can pin down π_0 uniquely. If $\left|\phi_\pi\right| < 1$, then

$$\pi(z) = \frac{\pi_0 - \frac{1+\sigma\phi_y}{\sigma} z\tilde{\zeta}(z) + \frac{1}{\sigma}\left(\tilde{\zeta}(z) - \tilde{\zeta}(0)\right)}{\left(1 - z\phi_\pi\right)} \tag{I.55}$$

is analytic on the unit disk for arbitrary finite values of π_0. If however $1 < \left|\phi_\pi\right|$ (or simply $1 < \phi_\pi$ restricting ourselves to positive values of ϕ_π), then $\pi(z)$ has a singularity on the unit disc at $z = 1/\phi_\pi$ that can be removed by setting the residue to zero $\lim_{z\to 1/\phi_\pi}\left(1 - z\phi_\pi\right)\pi(z) = 0$ to continue $\pi(z)$ as an analytic function over this singularity on the unit disk.

Appendix J. Proof Theorem 7

By the Wold theorem,[33] any stationary process can be represented as

$$X_t = \sum_{l=0}^{\infty} \theta_l \epsilon_{t-l} + \Xi_t, \text{ where } E\epsilon_t = 0 \text{ and } E\epsilon_t \epsilon'_{t+j} = 0,\ \forall j \neq 0 \tag{J.56}$$

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[33]See, e.g., Sargent (1987a, pp. 286–290), as well as Priestley (1981, pp. 756–758).

and Ξ_t is an orthogonal linearly deterministic process, forecastable perfectly from its own history. Starting with the indeterministic part,and inserting into (93)

$$0 = \sum_{j=0}^{n} \left[\sum_{l=0}^{\infty} \left(\sum_{i=0}^{min(p,l)} Q(i,j) \right) \theta_{l+j} \epsilon_{t-l} \right] + \sum_{j=1}^{m} \left[\sum_{l=0}^{\infty} \left(\sum_{i=0}^{min(p,l+j)} Q(i,j) \right) \theta_{l} \epsilon_{t-l-j} \right] \tag{J.57}$$

Using the definition of $\tilde{Q}(i,j)$ yields

$$0 = \sum_{j=0}^{n} \left[\sum_{l=0}^{\infty} \tilde{Q}(l,j) \theta_{l+j} \epsilon_{t-l} \right] + \sum_{j=1}^{m} \left[\sum_{l=0}^{\infty} \tilde{Q}(l+j,j) \theta_{l} \epsilon_{t-l-j} \right] \tag{J.58}$$

This must hold for all realizations of ϵ_t. Comparing coefficients yields

$$0 = \sum_{j=0}^{n} \tilde{Q}(l,j)\theta_{l+j} + \sum_{j=1}^{m} \tilde{Q}(l,j)\theta_{l-j} \tag{J.59}$$

a time-varying system of difference equations with initial conditions $\sum_{j=1}^{m}\theta_{-j} = 0$. But as $\tilde{Q}(p+i,j) = \tilde{Q}(p,j)$, $\forall i \geq 0$, the system of difference equations has constant coefficients, after and including p. This system can be written as (94) and coincides with Anderson's (2010) canonical form. If the solution to this system is unique, its stable solution can be written as

$$\theta_l = B \begin{bmatrix} \theta'_{l-m} & \dots & \theta'_{l-1} \end{bmatrix}', \ \forall l \geq p \tag{J.60}$$

The first p (block) equations—remembering the initial conditions—can be gathered into

$$\mathbf{Q} \begin{bmatrix} \theta'_0 & \dots & \theta'_{n+p-1} \end{bmatrix}' = 0 \tag{J.61}$$

giving $3p$ equations in $3(p+n)$ variables. (J.60) yields $3n$ more equations that deliver

$$\mathbf{B} \begin{bmatrix} \theta'_0 & \vdots & \theta'_{n+p-1} \end{bmatrix}' = 0 \tag{J.62}$$

stacking the two yields (95).[34]

The system (J.59) is homogenous. Thus, one stationary solution is given by $\theta_l = 0$, $\forall i$, the fundamental solution in the absence of exogenous driving forces. If (95) is invertible and if (94) is saddle-point stable, then this is the only stationary solution.

Only Ξ_t remains. Inserting it into (93), it follows that this can also be written as (94). If there is a unique solution in past values of Ξ_t, the solution can be written in the same form as (J.60), which must be zero when taken to its remote past from the stability of (J.60).

Appendix K. Example of Singular Information Structure in Theorem 7

The first condition theorem 7 requires that the model be determinate if the information rigidity were removed and the second requires that one can uniquely resolve the prediction errors, which would only fail to hold due to the non-singularity of the matrix $\begin{bmatrix} \mathbf{Q}' & \mathbf{B}' \end{bmatrix}'$. While this cannot be guaranteed due to the generality of the class of models specified in (93), there is nothing in the class of models to induce this matrix to be singular in general. Even if one should encounter a parameterization that leads to singularity, a minor perturbation of the model or its parameterization should generally lead to non-singularity.

[34]This extends Meyer-Gohde's (2010, p. 987) Equation (12) to Anderson's (2010) higher leads and lags.

A simple, univariate example will illustrate. Consider the following system

$$aE_t[\theta_{t+1}] = b\theta_t + cE_{t-1}[\theta_t] \tag{K.63}$$

In the absence of expectations, that is in the form of (94), the equation reduces to

$$a\theta_{t+1} = (b+c)\theta_t \tag{K.64}$$

which is saddle-point stable if $|\frac{b+c}{a}| > 1$. But the original equation does have expectations and, indeed, lagged expectations that need to be resolved. Taking expectations of (K.63) at the highest expectational lag (here $t-1$) yields

$$aE_{t-1}[\theta_{t+1}] = (b+c)E_{t-1}[\theta_t] \tag{K.65}$$

defining $\tilde{\theta}_{t-1} = E_{t-1}[\theta_t]$, inserting into the above and lagging forward yields

$$aE_t\left[\tilde{\theta}_{t+1}\right] = (b+c)\tilde{\theta}_t \tag{K.66}$$

an equation whose saddle-point properties are the same as (K.64). Thus, if $|\frac{b+c}{a}| > 1$, there is a unique stable solution. As the system is homogenous, this is $\tilde{\theta}_t = 0$. Recalling the definition of $\tilde{\theta}_t$ and inserting into (K.63) yields

$$0 = b\theta_t \tag{K.67}$$

Consider now the special case $b = 0$: the foregoing does not deliver a unique solution for θ_t, even though the condition for saddle-point stability, now $|\frac{c}{a}| > 1$, can still be fulfilled. Of course, $b = 0$ is a special case and it need not hold generally: hence, an isolated singularity.

Appendix L. Proof of Theorem 9

Take the IS equation (2) and express it in the frequency domain

$$y(z) = \frac{1}{z}(y(z) - y_0) - \sigma R(z) + \sigma\frac{1}{z}(\pi(z) - \pi_0) \tag{L.68}$$

do the same with the Taylor rule in (99)

$$R(z) = \phi_\pi z^{-j}\left(\pi(z) - \sum_{k=0}^{j-1}\pi_k z^k\right) + \phi_y z^{-m}\left(\tilde{y}(z) - \sum_{k=0}^{m-1}\tilde{y}_k z^k\right) \tag{L.69}$$

where $\tilde{y}(z) = (1-(1-\alpha)z)y(z)$. Now combine the two

$$\frac{1-z}{z}y(z) - \frac{1}{z}y_0 = \sigma\left(\phi_\pi z^{-j}\left(\pi(z) - \sum_{k=0}^{j-1}\pi_k z^k\right) + z^{-m}\phi_y\left(\tilde{y}(z) - \sum_{k=0}^{m-1}\tilde{y}_k z^k\right)\right) - \sigma\frac{1}{z}(\pi(z) - \pi_0) \tag{L.70}$$

collecting terms

$$\left(\phi_\pi z^{1-j} - 1\right)\pi(z) = \frac{1}{\sigma}(1-z)y(z) - \frac{1}{\sigma}y_0 - \phi_y z^{1-m}\left(z\tilde{y}(z) - \sum_{k=0}^{m-1}\tilde{y}_k z^k\right) + \phi_\pi z^{1-j}\sum_{k=0}^{j-1}\pi_k z^k + \pi_0 \tag{L.71}$$

Now recall that $y(z)$ follows from $\pi(\lambda z)$ and further dampened (as $0 < \lambda < 1$) inflation

$$y(z) = \frac{1}{\xi}\sum_{j=1}^{\infty}\frac{\lambda^j}{1-\lambda^j z}\pi(\lambda^j z) \tag{L.72}$$

Hence, given $\pi(\lambda^j z)$; $j > 0$, $y(z)$ and all $y_k \equiv \left(d^k y(z)/dz^k\right)|_{z=0}$ follow from (L.72).

Note that (L.71) defines $\pi(z)$ with roots $z:\phi_\pi z^{1-j}-1=0$. For a given root, call it $\overline{z^{(1)}}$, (L.71) implies roots for $\pi(\lambda^k z)$ as $z:\phi_\pi\left(\lambda^k z\right)^{1-j}-1=0 \Rightarrow \phi_\pi\lambda^{k(1-j)}z^{1-j}-1=0$. Corresponding to $\overline{z^{(1)}}$ is the root for $\pi(\lambda^k z)$, call it $\overline{\lambda^k z^{(1)}}$. So $\overline{\lambda^k z^{(1)}}$ solves $\phi_\pi\lambda^{k(1-j)}\overline{\lambda^k z^{(1)}}^{1-j}-1=0$ and $\overline{z^{(1)}}$ solves $\phi_\pi\overline{z^{(1)}}^{1-j}-1=0$. Inspection shows that the roots are related via $\overline{\lambda^k z^{(q)}}=\lambda^k\overline{z^{(q)}}$, for $q=1,2,..$ # of roots. Now (L.71) has $\tilde{y}(z)$ and $y(z)$ on the right hand side which, via (L.72) and the definition of $\tilde{y}(z)$, are linear functions of $\pi(\lambda^j z)$; $j>0$ and it follows that a root $\pi(z)$ on the left hand side, $z:\phi_\pi z^{1-j}-1=0$, corresponds to a root on the right hand side in the terms $\pi(\lambda^j z)$; $j>0$. That is, extending $\pi(z)$ by removing a singularity at a root $\overline{z^{(q)}}$ removes the corresponding singularity in $\pi(\lambda^k z)$ via $\pi(z)|_{z=\overline{z^{(q)}}}=\pi(\lambda^k z)|_{z=\overline{z^{(q)}}}$ which is evaluating $\pi(\lambda^k z)$ at its root $\overline{\lambda^k z^{(q)}}$ as $\lambda^k\overline{z^{(q)}}=\overline{\lambda^k z^{(q)}}$. Hence, eliminating roots inside the unit circle allows (L.71) to define $\pi(z)$ as an analytic function - and thus also $y(z)$ via (L.72) - over the unit disk. That is, the long run verticality of the Phillips curve (L.72) or independence of $y(z)$ from $\pi(z)$ on the unit circle translates the singularities in $\pi(z)$ to singularities in $y(z)$ - via $\pi(\lambda^k z)$. The elimination of singularities follows thus only via the independent consideration of singularities in $\pi(z)$.

Rewritting (L.71)

$$\left(\phi_\pi z^{1-j}-1\right)\pi(z)=\phi_\pi z^{1-j}\sum_{k=0}^{j-1}\pi_k z^k-\pi_0+\text{t.i.d.} \tag{L.73}$$

where t.i.d. refers to “terms independent of determinacy" following the discussion above. This allows us to easily declinate the problem into the number of roots.

For $j<1$, the summation on the right hand side is empty

$$\left(\phi_\pi z^{1-j}-1\right)\pi(z)=\pi_0+\text{t.i.d.} \tag{L.74}$$

therefore only one constant, π_0, needs to be determined. That is, the polynomial $\phi_\pi z^{1-j}-1=0$ must have one and only one z inside the unit circle for the system to be determinate, for π_0 to be set to remove the singularity at the root inside the unit circle so that $\pi(z)$ (and hence $y(z)$) is an analytic function over the unit disk. If there are no roots inside the unit circle, then π_0 cannot be pinned down and the system is indeterminate. If there is more than one root inside the unit circle, then there are not enough constants that can be set to eliminate the singularities to render $\pi(z)$ (and hence $y(z)$) analytic functions over the entire unit disk. The roots are given by

$$z=\left(\frac{1}{\phi_\pi}\right)^{\frac{1}{1-j}} \tag{L.75}$$

If $1<\phi_\pi$, then all $1-j$ roots are inside the unit circle. If $0<\phi_\pi<1$, then all $1-j$ roots are outside the unit circle. This gives the following

$$\begin{cases}\text{for } j=0, & 1-j=1 \text{ root inside the unit circle if and only if } 1<\phi_\pi\\ \text{for } j<0, & 1-j>1 \text{ roots inside/outside the unit circle if } 1<\phi_\pi\ /\ 0<\phi_\pi<1\end{cases} \tag{L.76}$$

For $j\geq 1$, (L.71) becomes

$$\left(\phi_\pi-z^{j-1}\right)\pi(z)=\phi_\pi\sum_{k=0}^{j-1}\pi_k z^k+z^{j-1}\pi_0+\text{t.i.d.} \tag{L.77}$$

and therefore j constants, $\{\pi_k\}_{k=0,1,\ldots,j-1}$, need to be determined. That is, the polynomial $\phi_\pi-z^{j-1}=0$ must have j roots inside the unit circle for the system to be determinate, for $\{\pi_k\}_{k=0,1,\ldots,j-1}$ to be set to remove the singularity at the roots inside the unit circle so that $\pi(z)$ (and hence $y(z)$) is an analytic function over the unit disk. If there are fewer roots inside the unit circle, then not all of $\{\pi_k\}_{k=0,1,...,j-1}$ cant be pinned down and the system is indeterminate. If there are more than j roots inside the unit circle, then there are not enough constants that can be set to eliminate the singularities to render $\pi(z)$ (and hence $y(z)$) analytic functions over the entire unit disk. The polynomial $\phi_\pi - z^{j-1} = 0$ is of order $j-1$ and, hence, has $j-1 < j$ roots following from the fundamental theorem of algebra. That is

$$\begin{cases} \text{for } j \geq 1, & \text{less than } j \text{ roots inside the unit circle} \end{cases} \tag{L.78}$$

Summarizing over the cases yields theorem 9 and the lower panel of figure 1.

Appendix M. Proof of Theorem 10

Rouché's theorem, also at the foundation of familiar Schur-Cohn (Woodford, 2003; Lubik and Marzo, 2007) and Jury conditions, will be used in the following and is worth repeating here

Theorem 14 (Rouché's Theorem)**.** *Let f and g be holomorphic in an open region containing the closure of the unit disk, such that g does not vanish on the unit circle. If $|f(z)| < |g(z)|$ on the unit circle, then f and $f+g$ have the same number of zeros, counting multiplicities, inside the unit circle.*

Proof. See Ahlfors (1979, pp. 152-154) □

The Taylor rule in (101) in the frequency domain is

$$\left(1-\rho_R z\right) R(z) = \left(1-\rho_R\right)\left[\phi_\pi z^{-j}\left(\pi(z) - \sum_{k=0}^{j-1} \pi_k z^k\right) + \phi_y z^{-m}\left(\tilde{y}(z) - \sum_{k=0}^{m-1} \tilde{y}_k z^k\right)\right] \tag{M.79}$$

where again $\tilde{y}(z) = (1-(1-\alpha)z)y(z)$. Combining this with the IS equation (L.68) then gives

$$\frac{1-z}{z} y(z) - \frac{1}{z} y_0 = \sigma \frac{(1-\rho_R)}{(1-\rho_R z)}\left[\phi_\pi z^{-j}\left(\pi(z) - \sum_{k=0}^{j-1} \pi_k z^k\right) + \phi_y z^{-m}\left(\tilde{y}(z) - \sum_{k=0}^{m-1} \tilde{y}_k z^k\right)\right] - \sigma \frac{1}{z}(\pi(z) - \pi_0) \tag{M.80}$$

collecting terms

$$\left(1-\rho_R z - \left(1-\rho_R\right)\phi_\pi z^{1-j}\right)\pi(z) \tag{M.81}$$

$$= \left(1-\rho_R z\right)\pi_0 - \left(1-\rho_R\right)\phi_\pi z^{1-j} \sum_{k=0}^{j-1} \pi_k z^k - \frac{1-\rho_R z}{\sigma} y_0 - \left(1-\rho_R\right)\phi_y z^{1-m} \sum_{k=0}^{m-1} \tilde{y}_k z^k \tag{M.82}$$

$$+ \left[\left(1-\rho_R z\right)(1-z)\frac{1}{\sigma} + \left(1-\rho_R\right)\phi_y z^{1-m}(1-(1-\alpha)z)\right] y(z) \tag{M.83}$$

Now recall that $y(z)$ follows from $\pi(\lambda z)$ and further dampened (as $0 < \lambda < 1$) inflation, see (L.72), hence, $y(z)$ and all $y_k \equiv \left(d^k y(z)/dz^k\right)|_{z=0}$ follow from (L.72) given $\pi(\lambda^j z)$; $j > 0$.

Note that (M.81) defines $\pi(z)$ with roots $z : 1-\rho_R z - \left(1-\rho_R\right)\phi_\pi z^{1-j} = 0$. Following the proof of theorem 9 above, extending $\pi(z)$ by removing a singularity at a root $\overline{z^{(q)}}$ removes the corresponding singularity in $\pi(\lambda^k z)$ via $\pi(z)|_{z=\overline{z^{(q)}}} = \pi(\lambda^k z)|_{z=\overline{z^{(q)}}}$ which is evaluating $\pi(\lambda^k z)$ at its root $\overline{\lambda^k z^{(q)}}$ as $\lambda^k \overline{z^{(q)}} = \overline{\lambda^k z^{(q)}}$. Hence, eliminating roots inside the unit circle allows (M.81) to define $\pi(z)$ as an analytic function - and thus also $y(z)$ via (L.72) - over the unit disk. That is, the long run verticality of the Phillips curve (L.72) or independence of $y(z)$ from $\pi(z)$ on the unit circle translates the singularities in $\pi(z)$ to singularities in

$y(z)$ - via $\pi(\lambda^k z)$. The elimination of singularities follows thus only via the independent consideration of singularities in $\pi(z)$.

Rewritting (M.81)

$$\left(1-\rho_R z-\left(1-\rho_R\right)\phi_\pi z^{1-j}\right)\pi(z)=\left(1-\rho_R z\right)\pi_0-\left(1-\rho_R\right)\phi_\pi z^{1-j}\sum_{k=0}^{j-1}\pi_k z^k+\text{t.i.d.} \tag{M.84}$$

where t.i.d. refers to "terms independent of determinacy" following the discussion above. This allows us to easily declinate the problem into the number of roots.

For $j\le 1$, the right hand side is in π_0 (that is, the summation on the right hand side contains at most a term in π_0)

$$\left(1-\rho_R z-\left(1-\rho_R\right)\phi_\pi z^{1-j}\right)\pi(z)=\left[1-\rho_R z-\mathbf{1}_{j=1}\left(1-\rho_R\right)\right]\pi_0+\text{t.i.d.} \tag{M.85}$$

where $\mathbf{1}_{j=1}$ is the indicator function, equal to 1 if $j=1$ and 0 otherwise; therefore only one constant, π_0, needs to be determined. That is, the polynomial $1-\rho_R z-\left(1-\rho_R\right)\phi_\pi z^{1-j}=0$ must have one and only one z inside the unit circle for the system to be determinate, for π_0 to be set to remove the singularity at the root inside the unit circle so that $\pi(z)$ (and hence $y(z)$) is an analytic function over the unit disk. If there are no roots inside the unit circle, then π_0 cannot be pinned down and the system is indeterminate. If there is more than one root inside the unit circle, then there are not enough constants that can be set to eliminate the singularities to render $\pi(z)$ (and hence $y(z)$) analytic functions over the entire unit disk.

For $j=1$

For $j=1$, the polynomial becomes $1-\rho_R z-\left(1-\rho_R\right)\phi_\pi=0$ and the root is given by $z=\frac{1-(1-\rho_R)\phi_\pi}{\rho_R}$. Hence, the system is determinant if $\left|\frac{1-(1-\rho_R)\phi_\pi}{\rho_R}\right|<1$ or $1<\phi_\pi<\frac{1+\rho_R}{1-\rho_R}$ and indeterminant otherwise.

For $j=0$

For $j=0$, the polynomial becomes $1-\rho_R z-\left(1-\rho_R\right)\phi_\pi z=0$ and the root is given by $z=\frac{1}{\rho_R+(1-\rho_R)\phi_\pi}$. Hence, the system is determinant if $\left|\frac{1}{\rho_R+(1-\rho_R)\phi_\pi}\right|<1$ or $1<\phi_\pi$ and indeterminant otherwise.

For $j<0$

For $j<0$, the polynomial becomes $1-\rho_R z-\left(1-\rho_R\right)\phi_\pi z^k$ for $k=1-j>1$. To bound the number of zeros using Rouché's theorem, theorem 14 above, we will factor this polynomial to have the leading term in z^k monic and define its inverse polynomial. Accordingly, (M.85) can be factored as

$$-\left(1-\rho_R\right)\phi_\pi\left(z^k+\frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}z-\frac{1}{1-\rho_R}\frac{1}{\phi_\pi}\right)\pi(z)=\left[1-\rho_R z-\mathbf{1}_{j=1}\left(1-\rho_R\right)\right]\pi_0+\text{t.i.d.} \tag{M.86}$$

and the relevant polynomial becomes $z^k+\frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}z-\frac{1}{1-\rho_R}\frac{1}{\phi_\pi}$. Define $f(z)\equiv z^k$ and $g(z)\equiv\frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}z-\frac{1}{1-\rho_R}\frac{1}{\phi_\pi}$. The polynomial $f(z)$ has k zeros inside the unit circle (k zeros at the origin to be precise) and as

$$\min|f(z)|_{|z|=1}>\max|g(z)|_{|z|=1}\Rightarrow 1>\frac{1}{1-\rho_R}\frac{1}{\phi_\pi}\max\left|1-\rho_R z\right|_{|z|=1}\Rightarrow 1>\frac{1+\rho_R}{1-\rho_R}\frac{1}{\phi_\pi} \tag{M.87}$$

Then for $\phi_\pi>\frac{1+\rho_R}{1-\rho_R}$, the polynomial $f(z)+g(z)$ (our relevant polynomial $z^k+\frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}z-\frac{1}{1-\rho_R}\frac{1}{\phi_\pi}$ above) has the same number of roots as $f(z)$ inside the unit circle by virtue of Rouché's theorem, theorem 14 above. That is, the relevant polynomial has $k=1-j>1$ roots inside the unit circle which means there are too many roots inside the unit circle and hence there are not enough constants that can be set to eliminate the singularities to render $\pi(z)$ (and hence $y(z)$) analytic functions over the entire unit disk. We have nonexistence of a stationary solution.

Consider now the system using the reverse polynomial of $1-\rho_R z-\left(1-\rho_R\right)\phi_\pi z^k$, i.e., with $\tilde{z}\equiv 1/z$

$$\left(\tilde{z}^k-\rho_R\tilde{z}^{k-1}-\left(1-\rho_R\right)\phi_\pi\right)\pi(1/\tilde{z})=\left[\tilde{z}^k\left(1-\mathbf{1}_{j=1}\left(1-\rho_R\right)\right)-\rho_R\tilde{z}^{k-1}\right]\pi_0+\text{t.i.d.} \tag{M.88}$$

For determinacy, we must have one and only one z inside the unit circle which translates to all but one (that is $k-1$) $\tilde{z}$ inside the unit circle. Define $f(\tilde{z})\equiv\tilde{z}^k-\rho_R\tilde{z}^{k-1}=\tilde{z}^{k-1}\left(\tilde{z}-\rho_R\right)$. As $|\rho_R|<1$, $f(\tilde{z})$ has k zeros inside the unit circle (one at ρ_R and $k-1$ at the origin). Define as well $g(\tilde{z})\equiv-\left(1-\rho_R\right)\phi_\pi$. As $|g(\tilde{z})|=\left(1-\rho_R\right)\phi_\pi$ and $\min|f(\tilde{z})|_{|\tilde{z}|=1}=1-\rho_R$ it follows that

$$\min|f(\tilde{z})|_{|\tilde{z}|=1}>\max|g(\tilde{z})|_{|\tilde{z}|=1}\Rightarrow 1-\rho_R>1-\rho_R\phi_\pi\Rightarrow\phi_\pi<1 \tag{M.89}$$

Thus for $\phi_\pi<1$, the polynomial $f(\tilde{z})+g(\tilde{z})$ (our relevant polynomial $\tilde{z}^k-\rho_R\tilde{z}^{k-1}-\left(1-\rho_R\right)\phi_\pi$ above) has the same number of roots as $f(\tilde{z})$ inside the unit circle by virtue of Rouché's theorem, theorem 14 above. That is, the relevant polynomial has $k=1-j>1$ roots inside the unit circle which translates (as $\tilde{z}\equiv 1/z$) to no roots inside the unit circle for our original polynomial $1-\rho_R z-\left(1-\rho_R\right)\phi_\pi z^{1-j}$. Thus we have no singularities inside the unit circle that can be removed by pinning down the arbitrary constant π_0 and hence we have indeterminacy.

For $j>1$

For $j>1$, define $k=j-1>0$ and (M.84) becomes

$$\left(1-\rho_R z-\left(1-\rho_R\right)\phi_\pi z^{-k}\right)\pi(z)=\left(1-\rho_R z\right)\pi_0-\left(1-\rho_R\right)\phi_\pi z^{1-j}\sum_{i=0}^{k}\pi_i z^i+\text{t.i.d.} \tag{M.90}$$

where the right hand side is a function of $\pi_0,\pi_1,\ldots\pi_k$. Hence the system has $k+1$ coefficients to pin down and accordingly the polynomial $1-\rho_R z-\left(1-\rho_R\right)\phi_\pi z^{-k}$ must have $k+1$ roots inside the unit circle for the system to be determinate, for $\{\pi_i\}_{i=0,1,\ldots,k}$ to be set to remove the singularity at the roots inside the unit circle so that $\pi(z)$ (and hence $y(z)$) is an analytic function over the unit disk. If there are fewer roots inside the unit circle, then not all of $\{\pi_i\}_{i=0,1,\ldots,k}$ cant be pinned down and the system is indeterminate. If there are more than $k+1$ roots inside the unit circle, then there are not enough constants that can be set to eliminate the singularities to render $\pi(z)$ (and hence $y(z)$) analytic functions over the entire unit disk. Rewritting the polynomial as $\left(z^k-\rho_R z^{k+1}-\left(1-\rho_R\right)\phi_\pi\right)z^{-k}$ and hence determinacy requires $z^k-\rho_R z^{k+1}-\left(1-\rho_R\right)\phi_\pi$ to have $k+1$ roots inside the unit circle. The polynomial $z^k-\rho_R z^{k+1}-\left(1-\rho_R\right)\phi_\pi$ is of order $k+1$ and, hence, has $k+1$ roots following from the fundamental theorem of algebra and therefore cannot have more than $k+1$ roots. Therefore, the system will be either determinate or indeterminate.

Beginning with $z^k-\rho_R z^{k+1}-\left(1-\rho_R\right)\phi_\pi$ and defining $f(z)\equiv z^k-\rho_R z^{k+1}$ and $g(z)\equiv-\left(1-\rho_R\right)\phi_\pi$, $\min|f(z)|_{|z|=1}=1-\rho_R$ and $\max|g(z)|_{|z|=1}=\left(1-\rho_R\right)\phi_\pi$. Noticing that $|\rho_R|<1$, $f(z)$ has only k zeros inside the unit circle (k at the origin but one at $1/\rho_R$) and

$$\min|f(z)|_{|z|=1}>\max|g(z)|_{|z|=1}\Rightarrow 1-\rho_R>\left(1-\rho_R\right)\phi_\pi \tag{M.91}$$

Then for $\phi_\pi<1$, the polynomial $f(z)+g(z)$ (our relevant polynomial $z^k-\rho_R z^{k+1}-\left(1-\rho_R\right)\phi_\pi$ above) has the same number of roots as $f(z)$ inside the unit circle by virtue of Rouché's theorem, theorem 14 above. That is, the relevant polynomial has only k roots inside the unit circle which means there are too few singularities inside the unit circle that can be removed to pin down all the constants $\{\pi_i\}_{i=0,1,\ldots,k}$. We have indeterminacy or nonuniqueness of the stationary solution.

As above, consider now the reverse polynomial with $\tilde{z} \equiv 1/z$

$$\tilde{z} - \rho_R - (1-\rho_R)\phi_\pi \tilde{z}^{k+1} \Rightarrow -(1-\rho_R)\phi_\pi \left(\tilde{z}^{k+1} - \frac{1}{1-\rho_R}\frac{1}{\phi_\pi}\tilde{z} + \frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}\right) \tag{M.92}$$

For determinacy, we must have $k+1$ roots in z inside the unit circle which translates to zero roots in $\tilde{z}$ inside the unit circle. Define $f(\tilde{z}) \equiv \tilde{z}^{k+1}$, and $f(\tilde{z})$ has $k+1$ zeros inside the unit circle (all at the origin). Define as well $g(\tilde{z}) \equiv -\frac{1}{1-\rho_R}\frac{1}{\phi_\pi}\tilde{z} + \frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi} = \frac{1}{1-\rho_R}\frac{1}{\phi_\pi}\left(\rho_R - \tilde{z}\right)$. As $|f(\tilde{z})|_{|\tilde{z}|=1} = 1$ and $\max|g(\tilde{z})|_{|\tilde{z}|=1} = \frac{1+\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}$, it follows that

$$\min|f(\tilde{z})|_{|\tilde{z}|=1} > \max|g(\tilde{z})|_{|\tilde{z}|=1} \Rightarrow 1 > \frac{1+\rho_R}{1-\rho_R}\frac{1}{\phi_\pi} \Rightarrow \frac{1+\rho_R}{1-\rho_R} < \phi_\pi \tag{M.93}$$

Thus for $\frac{1+\rho_R}{1-\rho_R} < \phi_\pi 1$, the polynomial $f(\tilde{z}) + g(\tilde{z})$ (our relevant polynomial $-(1-\rho_R)\phi_\pi\left(\tilde{z}^{k+1} - \frac{1}{1-\rho_R}\frac{1}{\phi_\pi}\tilde{z} + \frac{\rho_R}{1-\rho_R}\frac{1}{\phi_\pi}\right)$ above) has the same number of roots as $f(\tilde{z})$ inside the unit circle by virtue of Rouché's theorem, theorem 14 above. That is, the relevant polynomial has $k+1$ roots inside the unit circle which translates (as $\tilde{z} \equiv 1/z$) to no roots inside the unit circle for our original polynomial $1 - \rho_R z - (1-\rho_R)\phi_\pi z^{-k}$. Thus we have no singularities inside the unit circle that can be removed by pinning down the arbitrary constants $\{\pi_i\}_{i=0,1,\ldots,k}$ and hence we have indeterminacy.

Appendix N. Determinacy Bounds in Table 2

N.1. *Determinacy bounds for the sticky price model with a forward-looking rule featuring a change in output*

Consider the sticky price model, given by (8), (2) and the following Taylor rule:

$$R_t = \phi_\pi E_t \pi_{t+1} + \Delta y_{t+1} \tag{N.94}$$

We substitute the policy rule into the IS equation (2) and put the system involving the two endogenous variables y_t, π_t in the following form:

$$E_t x_{t+1} = c + A x_t \tag{N.95}$$

where $x_t = [y_t, \pi_t]', c = 0$ and

$$A = \begin{bmatrix} -\frac{\sigma(1-\phi_\pi)}{1-\sigma\phi_y} & \frac{\beta(1+\sigma\phi_y)+\kappa\sigma(1-\phi_\pi)}{\beta(1-\sigma\phi_y)0} \\ 1/\beta & -\kappa/\beta \end{bmatrix}. \tag{N.96}$$

The characteristic equation of a 2×2 system matrix A is given by $p(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$. Both roots of the characteristic equation lie outside the unit circle if and only if (see LaSalle, 1986, p.28):

$$|\det(A)| > 1 \qquad \text{and} \qquad |\mathrm{tr}(A)| < 1 + \det(A),$$

where

$$\det(A) = -\frac{(1-\sigma\phi_y)}{\beta(1-\sigma\phi_y)} \tag{N.97}$$

and

$$\mathrm{tr}(A) = -\frac{\sigma(1-\phi_\pi}{\beta(1-\sigma\phi_\pi} - \frac{\kappa}{\beta} \tag{N.98}$$

Over the admissible parameter range, the determinant is strictly above one, if $1/\sigma < \phi_y$, so that the first condition holds. The right-hand-side of the second condition implies that $1+\phi_y(1+\beta+\kappa)+\frac{1+\kappa+\beta}{\sigma} < \phi_\pi$, while the left-hand-side leads to $\phi_\pi < 1+\frac{\kappa+\beta}{\sigma}-\phi_y(1+\kappa+\beta)$ which provides the set of the necessary and sufficient conditions for a unique equilibrium.

N.2. *Determinacy bounds for the sticky information model with a forward-looking rule*

Consider the sticky information model, given by (1), (85) and the following Taylor rule:

$$R_t = \phi_\pi E_t \pi_{t+1} \tag{N.99}$$

Following theorem 10 case (5), the model has a unique, stable equilibrium if and only if

$$1 < \phi_\pi < 1 \tag{N.100}$$

which of course is never true, such that

$$\phi_\pi = \varnothing. \tag{N.101}$$

As determinacy in the model with a forward-looking interest rate is independent of output gap, the result holds also true for other Taylor rules featuring output gap dated at any point in time, i.e. for $R_t = \phi_\pi E_t \pi_{t+1} + y_t$, $R_t = \phi_\pi E_t \pi_{t+1} + y_{t+1}$ and $R_t = \phi_\pi E_t \pi_{t+1} + \Delta y_{t+1}$.

N.3. *Determinacy bounds for the sticky information model with a backward-looking rule*

Consider the sticky information model, given by (1), (85) and the following Taylor rule:

$$R_t = \phi_\pi E_t \pi_{t-1} \tag{N.102}$$

Following theorem 10 case (1), the model features indeterminacy if $\phi_\pi < 1 \; \forall \; j$. Further, according to case (3) the model equilibrium is however nonexistent if $1 < \phi_\pi$, $j = -1$, such that

$$\phi_\pi = \varnothing. \tag{N.103}$$

As these results are independent of output gap, they hold true for other Taylor rules featuring output gap dated at any point in time, i.e. for $R_t = \phi_\pi E_t \pi_{t-1} + y_t$ and $R_t = \phi_\pi E_t \pi_{t-1} + y_{t-1}$.

Appendix O. Proof of Theorem 11

For the frictionless case, see Davig and Leeper (2007). For the finite informational inertia case (78), begin with the one period case from Lucas (1973) in (10): take $t-1$ expectations of (78) to yield $E_{t-1}[y_t] = 0$, of the IS equation (2) to yield $\tilde{R}_{t-1} \equiv E_{t-1}[R_t] = E_{t-1}[E_t[\pi_{t+1}]]$, and of (102) to yield $\tilde{R}_{t-1} = E_{t-1}\left[\phi_\pi(s_t)\pi_t\right]$. Lagging forward and combining yields

$$E_t\left[\phi_\pi(s_{t+1})\pi_{t+1}\right] = E_t[E_{t+1}[\pi_{t+2}]] \tag{O.104}$$

or

$$0 = E_t\left[\phi_\pi(s_{t+1})\pi_{t+1} - E_{t+1}[\pi_{t+2}]\right] = \phi_\pi(s_{t+1})\pi_{t+1} - E_{t+1}[\pi_{t+2}] + \eta_{t+1} \tag{O.105}$$

where η_{t+1} is a forecast error . Now following Davig and Leeper (2007) we acknowledge the regime dependence of the foregoing as

$$\begin{bmatrix} \phi_{\pi,1} & 0 \\ 0 & \phi_{\pi,2} \end{bmatrix} \begin{bmatrix} \pi_{1,t+1} \\ \pi_{2,t+1} \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} E_t \pi_{1,t+2} \\ E_t \pi_{2,t+2} \end{bmatrix} + \begin{bmatrix} \eta_{t+1} \\ \eta_{t+1} \end{bmatrix} \tag{O.106}$$

inverting the leading matrix in the foregoing gives

$$\overline{\pi}_{t+1} = M E_{t+1} \overline{\pi}_{t+2} + \overline{\phi_\pi}^{-1} \overline{\eta}_{t+1} \tag{O.107}$$

we can solve forward using Blanchard (1979) if and only if $|eig(M)| < 1$ which are the conditions given in the theorem and by Davig and Leeper (2007). To see that this pins down the forecast error (which ought to be zero in the absence of exogenous driving forces) note first that the forward solution gives

$$\overline{\pi}_{t+1} = \overline{\phi_\pi}^{-1} \overline{\eta}_{t+1} \tag{O.108}$$

or

$$\pi_{i,t+1} = \phi_{\pi,i}^{-1} \overline{\eta}_{t+1} \tag{O.109}$$

and $E_t \pi_{i,t+1} = 0$ for each i individually, hence $E_t \pi_{t+1} = 0$. From the supply equation (10) then, $y_{i,t} = \alpha \pi_{i,t}$ for the realized regime i in t. IS demand (2) gives $y_{i,t} = -\sigma R_{i,t}$ and the monetary policy rule $R_{i,t} = \phi_{\pi,i} \pi_{i,t} + \phi_{y,i} y_{i,t}$. We combine the three to get

$$\left(1 + \sigma \phi_{y,i}\right) \alpha \pi_{i,t} = -\sigma \phi_{\pi,i} \pi_{i,t} \to \left[\left(1 + \sigma \phi_{y,i}\right) \alpha + \sigma \phi_{\pi,i}\right] \pi_{i,t} \tag{O.110}$$

as long as $\left(1 + \sigma \phi_{y,i}\right) \alpha + \sigma \phi_{\pi,i} \neq 0$ - a singular information structure that we abstract from as stated in the theorem -[35] we conclude that

$$\pi_{i,t} = 0 \Rightarrow R_{i,t}, y_{i,t} = 0 \tag{O.111}$$

and the forecast error is indeed zero.

The general case in (78) is a straightforward extension of the above and the proof of theorem 7. With the higher expectational horizon at which the Phillips curve becomes vertical, we need to take $t-q$ expectations of (78) to yield $E_{t-q}[y_t] = 0$, which delivers

$$E_t \left[\phi_\pi(s_{t+q}) \pi_{t+q}\right] = E_t \left[E_{t+q}\left[\pi_{t+q+1}\right]\right] \tag{O.112}$$

or

$$0 = E_t \left[\phi_\pi(s_{t+q}) \pi_{t+q} - E_{t+q}\left[\pi_{t+q+1}\right]\right] = \phi_\pi(s_{t+q}) \pi_{t+q} - E_{t+q}\left[\pi_{t+q+1}\right] + \eta_{t+q} \tag{O.113}$$

The same homogenous structure as above yielding the same M and hence the same condition $|eig(M)| < 1$ as above and in the theorem. The complication compared with above is that instead of two one step forecast errors for each realized regime, there are now sequences of forecast errors out to the horizon q conditional on the realized regimes. Appealing to the homogeneity of the equations in the absence of exogenous driving forces and abstracting from singularities in the information structure - see the explicit revelation above as

[35]See section K for another example. Here we have two such singularities indexed by the realized regime.

an example in the one period case - the zero solution for all variables and forecast errors is thus the unique stable solution if and only if $|eig(M)| < 1$.

The sticky information and imperfect common knowledge supply follow as extensions of the results above and in theorems 6 and 4 - the homogenous structure is the same as above and hence the same condition $|eig(M)| < 1$ is relevant for determinacy. If it is fulfilled, the homogeneity of the problem suffices to establish the zero solution for all variables and forecast errors as the unique stable solution as we by assumption rule out informational singularities that would deliver a singularity in one (or more) equations needed to resolve the sequences of forecasting and prediction errors. As noted in the main text, being able to analytically characterize the determinacy conditions does not mean that we are able to analytically solve for the response of forecasting and prediction errors and variables to the realization of an inhomogenous shock in these infinite dimensional cases.

Appendix P. Proof for Theorem 12

For the finite informational inertia case (78), begin with the one period case from Lucas (1973) in (10): take $t-1$ expectations of (78) to yield $E_{t-1}[y_t] = 0$ and of the IS equation (2) to yield

$$\tilde{R}_{t-1} \equiv E_{t-1}[R_t] = E_{t-1}[E_t[\pi_{t+1}]] = E_{t-1}[\tilde{\pi}_t] \tag{P.114}$$

Combining (104) and (105) and taking $t-1$ expectations gives

$$\tilde{b}_{t-1} \equiv E_{t-1}[b_t] = \left(\beta^{-1} - \phi_d\right) b_{t-1} - \beta^{-1}\tilde{\pi}_{t-1} + \tilde{R}_{t-1} \tag{P.115}$$

The $t-1$ expectations of the Taylor rule (1) gives $\tilde{R}_{t-1} = \phi_\pi \tilde{\pi}_{t-1} + \phi_y E_{t-1}[y_t]$. Recall from Lucas (1973) in (10) that $E_{t-1}[y_t] = 0$, hence

$$\tilde{R}_{t-1} = \phi_\pi \tilde{\pi}_{t-1} \tag{P.116}$$

Inserting this into (P.115) gives

$$\tilde{b}_{t-1} = E_{t-1}[b_t] = \left(\beta^{-1} - \phi_d\right) b_{t-1} + \left(\phi_\pi - \beta^{-1}\right)\tilde{\pi}_{t-1} \tag{P.117}$$

and solving for $\tilde{\pi}_{t-1}$ yields

$$\tilde{\pi}_{t-1} = \frac{1}{\phi_\pi - \beta^{-1}}\tilde{b}_{t-1} - \frac{\beta^{-1} - \phi_d}{\phi_\pi - \beta^{-1}} b_{t-1} \tag{P.118}$$

Lagging (P.117) forward and taking $t-1$ expectations gives

$$E_{t-1}\left[\tilde{b}_t\right] = \left(\beta^{-1} - \phi_d\right)\tilde{b}_{t-1} + \left(\phi_\pi - \beta^{-1}\right) E_{t-1}[\tilde{\pi}_t] \tag{P.119}$$

Combining with the Fisher equation and the Taylor rule gives

$$E_{t-1}\left[\tilde{b}_t\right] = \left(\beta^{-1} - \phi_d\right)\tilde{b}_{t-1} + \left(\phi_\pi - \beta^{-1}\right)\phi_\pi \tilde{\pi}_{t-1} \tag{P.120}$$

Inserting (P.118) for $\tilde{\pi}_{t-1}$ gives

$$E_{t-1}\left[\tilde{b}_t\right] = \left(\beta^{-1} - \phi_d + \phi_\pi\right)\tilde{b}_{t-1} - \left(\beta^{-1} - \phi_d\right)\phi_\pi b_{t-1} \tag{P.121}$$

which determines the dynamics of debt. Inspection proves this has the same homogenous structure as equation (106) in the main text and, hence the same determinacy properties. As in the proof of theorem

11, the general case in (78) is a straightforward extension of the above and the proof of theorem 7 and the sticky information and imperfect common knowledge supply as follow extensions of the results above and in theorems 6 and 4. It is again the homogenous structure of the problem - i.e., abstracting from shocks and prediction forecasting errors - that is relevant for determinacy.